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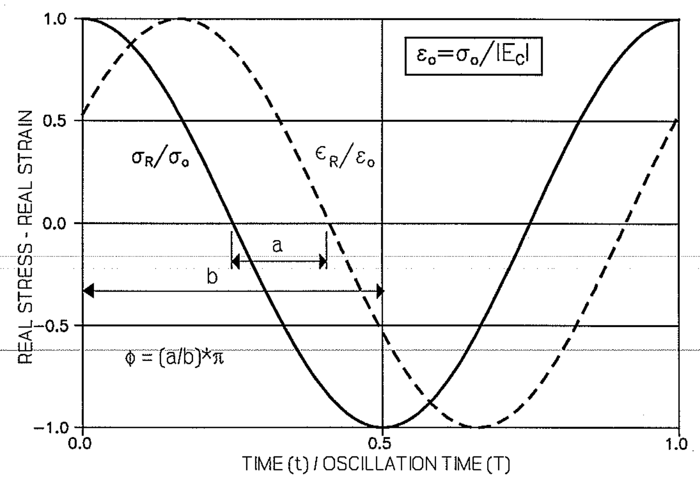

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Abstract

A method is explained by which experimental vibration analysis of systems can be generalized to become a useful analytical tool also in the research on materials rheology. When the method is used on data obtained from modern experimental vibration analysis of systems (using Brüel & Kjær type 2035 for example) rheological information of materials can be detected continuously in the time range from a very small fraction of a second to a couple of minutes. This range is very important from a materials science point of view. Basic mechanisms can be detected or estimated which control the materials long-term behavior such as creep, relaxation, and damping for example.

Examples from practice on building materials where such information are valuable are: Quality control in materials production, non-destructive testing of materials such as concrete and wood, detection and quantification of progressing materials decomposition due to freezing or salt infection, change of materials behavior with respect to temperature such as in cement- and asphaltic concretes, change of composite materials behavior in general with respect to mixture variations. Theoretically there is no problem in increasing the working range of an experimental vibration analysis from the few minutes previously mentioned to longer times. For long-time studies on the rheology of materials it is, however, more appropriate to combine experimental vibration analysis with alternate methods suggested in the paper.

1. Introduction

Very accurate vibration studies on structures can be made to day with advanced experimental vibration analysis equipment (ex 1). Such equipment, however, can be used also as a powerful tool in material research. Rheological (viscoelastic) properties of materials can be determined from modern experimental vibration analysis on test specimens (systems) made of the material considered*).

In principle all material properties which can be obtained by modern vibration analysis can also be obtained by the well-known resonance frequency method (2). The former method, however, is very superior with respect to data utilization and practical handling. All vibration data are used and only one test specimen is required. In a resonance frequency analysis only data at resonance frequencies can be used which means that a large number of different test specimens have to be prepared. Another advantage of using modern vibration analysis is that the results obtained are more reliable when materials are tested with strong internal damping.

The theoretical basics of rheological material properties determined from experimental vibration analysis of systems are presented in this paper as they are lectured (ex 3,4) by the author in Course 6110: Material Mechanics and used (ex 5) at the Building Materials Laboratory, Technical University of Denmark:

- The stress-strain behavior of viscoelastic materials is related theoretically to the stress-strain behavior of elastic materials.
- The force-deflection behavior of viscoelastic systems is related theoretically to the force-deflection behavior of elastic systems and viscoelastic material properties.
- The viscoelastic material properties (such as complex stiffness, creep and relaxation) are deduced from the (experimentally obtained) force-deflection relationship of the viscoelastic system.

Material properties and system properties are stiffnesses defined as stress/strain [Pa] and force/deflection [N/m] respectively. The material property deduced by the method presented is the one which controls the materials influence on the test systems behavior. Example from this paper: Material properties to determine are those which relate *uni-directional stress to strain in viscoelastic materials*. The obvious choice of test system for this purpose is an axially loaded bar.

The method presented can easily be modified to consider other material properties (associated with bending or shear for example) by using test systems appropriately designed for this purpose.

*) An example is presented by N. Johan Wismer in the second paper of this DSM-publication.

1.1 Reader's guidance

The paper is organized according to the steps outlined in previous section. Perspectives of the method presented are discussed in the final section of the paper. Three auxiliary sections are added separately at the end of the paper:

Appendix A: Useful expressions on the handling of complex numbers are presented in this section. No further information on complex numbers are needed in reading the theoretical sections of the paper. For application of the theory computer math-programs are needed with complex number facilities.

Appendix B: Some information on the rheology of basic material models are presented in this section. They are useful when training the method presented in the paper - and also when the results obtained by the method are evaluated.

Appendix C: An elastic analysis is made on the bar system applied in this paper. The results obtained might as well be reproduced directly from the literature on elastic vibrations of structures. A detailed analysis, however, may be helpful in a more deep understanding of the main subject considered in the paper, systems versus materials.

1.2 Notations

Symbols frequently used in the paper are explained in the following list. Others are explained as they appear in the text.

	Materials
σ, ε	Stress, strain
$E = E_{DYN}$	Elastic stiffness (Dynamic Young's modulus) (from high speed test)
$J = 1/E$	Elastic flexibility
c(t), r(t)	Creep function, relaxation function
τ	Relaxation time
b	Creep power
$E_C = E_R + iE_I$	Complex stiffness
i	complex unit
E_R, E_I	Real part and imaginary part respectively of E_C
$J_C = 1/E_C$	Complex flexibility
$\lvert E_C \rvert$, $\lvert J_C \rvert$	Num. complex stiffness, num. complex flexibility
ϕ, $\tan\phi$	Loss angle, loss tangent
	System (bar fixed at one end)
l	Length
A	Cross-section area
ρ	Specific mass
μ	Mass per unit length of bar (= $A\rho$)
m	Mass of bar (= $lA\rho = l\mu$)

M	Discrete mass at free end
P, u	Force at end of bar, deflection
S	$= P/u$(end of bar), Elastic stiffness
$H = 1/S$	Elastic flexibility
$S_C = S_R + iS_I$	Complex stiffness (analogue to elastic S)
i	complex unit
S_R, S_I	Real part and imaginary part respectively of S_C
$H_C = 1/S_C$	Complex flexibility
$\lvert S_C \rvert$, $\lvert H_C \rvert$	Num. complex stiffness, num. complex flexibility
δ, $\tan\delta$	Loss angle, loss tangent
ω_n	Resonance frequency
	Others
t	Time
T	Oscillation time
$\omega = 2\pi/T$	Angular frequency
$f = 1/T = \omega/(2\pi)$	Frequency (conventional)

2. Materials

2.1 Elastic material

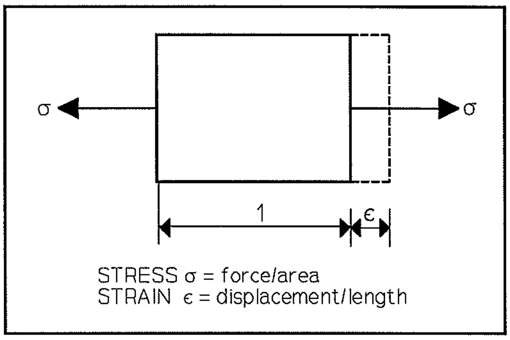

Figure 1. Stress on, and strain of elastic or viscoelastic material considered in this paper.

Stress and strain in so-called linear-elastic materials are related by the well-known Hooke's law presented in Equation 1. This relationship can be expressed with materials stiffness (E) or with materials flexibility (J = 1/E) as shown in Equation 1. Normally the former version is used. In subsequent text the terms elastic and linear-elastic are used synonymously.

Hooke's law does not depend on time which, for example, means that $\varepsilon = \sigma(t)/E$ always applies. This simple stress-strain-time relationship does not apply for the so-called viscoelastic materials subsequently considered.

$$\varepsilon = J\sigma \quad ; \quad \sigma = E\varepsilon \quad ; \quad (\textit{flexibility } J = 1/E) \tag{1}$$

2.2 Viscoelastic material

2.2.1 Creep and relaxation

Viscoelastic materials exhibit time dependent behavior even if subjected to constant loads. Creep functions (strain under constant load) and relaxation functions (stress under constant strain) are important characteristics for the behavior of viscoelastic materials.

An important class of viscoelastic materials are the linear-viscoelastic materials. They have creep functions which vary proportional to constant load used in creep experiment - and relaxation functions which vary proportional to constant strain used in relaxation experiment. The elastic materials previously considered are included in this description as a the special class of viscoelastic materials which have creep functions and relaxation functions which do not vary with time. The viscoelastic materials considered in this paper are of the linear type. In subsequent text the terms viscoelastic and linear-viscoelastic are used synonymously.

Creep functions c(t) are used in the meaning, "strain detected at constant stress = 1". In a similar way relaxation functions r(t) are used in the meaning, "stress detected at constant strain = 1". In this definition of creep function and relaxation function elastic materials are included with c(t) ≡ 1/E and r(t) ≡ E. Typical creep functions and relaxation functions look like the examples shown in Figures 2 and 3.

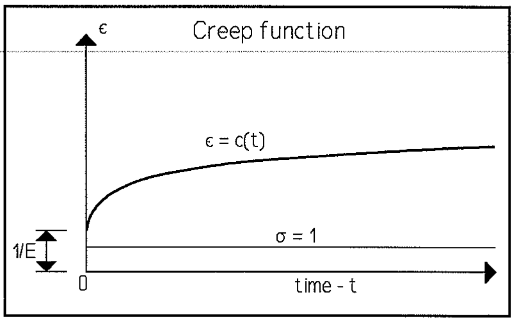

Figure 2. Creep function is strain of material subjected to a constant stress of magnitude 1.

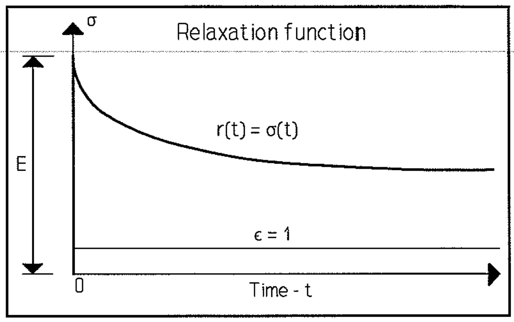

Figure 3. Relaxation function is stress in material subjected to a constant strain of magnitude 1.

Creep functions and relaxation functions for a number of material models are presented in Appendix A at the end of the paper. Of special interest in the context of building materials is the Power law creep expression considered separately in Section 2.2.5.

Note: *Materials with "higher" creep functions have "lower" relaxation functions. When subjected to constant loads they seem to become "softer" reducing there original stiffness as time proceeds .*

2.2.2 Stress-strain relation

It can be shown (ex 6) that viscoelastic stress-strain relations can be formulated as presented in Equation 2. The following analogy to elasticity is observed: Integral operators based on creep functions (c(t)) and relaxation functions (r(t)) replace the elastic flexibility (1/E) and elastic stiffness (E) respectively in the elastic stress-strain relations (Equation 1). The so-called elastic-viscoelastic analogy subsequently considered is a consequence of this observation.

$$\varepsilon = \int_0^t c(t-\theta)\frac{d\sigma}{d\theta}d\theta \quad ; \quad \sigma = \int_0^t r(t-\theta)\frac{d\varepsilon}{d\theta}d\theta \qquad (2)$$

The stress-strain relations in Eqaution 2 are fully correlated, just as their elastic counterparts in Equation 1 are. The viscoelastic behavior of a material is completely defined through one of them. The creep function and the relaxation function are related through the following expression

$$\int_0^t c(t-\theta)\frac{dr(\theta)}{d\theta}d\theta = \int_0^t r(t-\theta)\frac{dc(\theta)}{d\theta}d\theta = 1 \qquad (3)$$

Elastic-viscoelastic analogy

The following elastic-viscoelastic analogy (sligthly re-formulated) was used implicitely in (7,8) and stated explicitely in (9).

> *A general stress-strain analysis of a linear-viscoelastic structure can be made by replacing the coefficient of elasticity in a corresponding linear-elastic analysis by its associated viscoelastic operator,*

$$\frac{1}{E} \Rightarrow \int_0^t c(t - \theta)\frac{d[\ ]}{d\theta}d\theta \quad or \quad E \Rightarrow \int_0^t r(t - \theta)\frac{d[\ ]}{d\theta}d\theta \qquad (4)$$

2.2.3 Harmonic stress variation

It can be shown (ex 6) that Equation 2 reduces as follows when stress varies harmonically like sine or cosine waves.

$$\varepsilon = \frac{\sigma}{E_C} \quad with \quad \sigma = \sigma_o \exp(i\omega t) = \sigma_o[\cos(\omega t) + i\ \sin(\omega t)] \qquad (5)$$

The complex stiffness E_C of the material is a quantity which depends on angular frequency ω of loading and the creep/relaxation properties of the material considered. E_C can be determined from Equation 6 reproduced from (10,11,12),

$$E_C = E^A(i\omega) \quad where \quad E^A(s) = \frac{1}{s\mathcal{L}[c(t)]} = s\mathcal{L}[r(t)]$$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt \quad (Laplace\ transformed\ with\ complex\ variable\ s) \qquad (6)$$

The intermediate quantity $E^A(s)$ in Equation 6 is the so-called analogue stiffness which, as shown in references just mentioned, can be used very efficiently in the general analysis of viscoelastic materials and structures.

Analogue stiffnesses, and thus complex stiffnesses, for a number of material models are presented in Appendix A. The complex stiffness associated with the Power law creep expression previously referred to is presented separately in Section 2.2.5.

Elastic-viscoelastic analogy

It comes from the analogy previously presented in Section 2.2.2 that the analogy observed between Equation 5 and its elastic counterpart in Equation 1 can be generalized as follows:

The analysis of a viscoelastic problem with sine (or cosine) varying load can be made by the theory of elasticity with Young's modulus replaced by its viscoelastic counterpart, namely the complex Young's modulus.

The complex stiffness can in general be written as presented in Equation 7 where $|E_C|$ and ϕ denote numerical complex stiffness and loss angle respectively.

$$
\begin{aligned}
&E_C = E_R + iE_I = |E_C|\exp(i\phi) && \textit{complex stiffness} \\
&|E_C| = \sqrt{E_R^2 + E_I^2} && \textit{numerical complex stiffness} \\
&\tan(\phi) = \frac{E_I}{E_R} && \textit{loss tangent}
\end{aligned}
\qquad (7)
$$

Strain and energy loss

The general strain solution associated with Equation 5 becomes

$$
\begin{aligned}
&\varepsilon = \varepsilon_o \exp(i(\omega t - \phi)) = \varepsilon_o[\cos(\omega t - \phi)) + i \sin(\omega t - \phi] \\
&\varepsilon_o = \frac{\sigma_o}{|E_C|}
\end{aligned}
\qquad (8)
$$

which illustrates that stress and strain vary in similar ways. Two phenomenons, however, clearly demonstrates the influence of creep. Strain is delayed by ϕ, and its magnitude is modified by a factor of $E/|E_C|$ relative to the elastic counterpart solution $\varepsilon = \sigma/E$ ($\phi = 0$).

The energy loss W per volume unit per load cycle is given by

$$
W = \int_0^T \sigma \frac{d\varepsilon}{dt} dt = \frac{\pi\sigma_o^2}{|E_C|} \frac{\tan(\phi)}{\sqrt{1 + \tan^2(\phi)}} \quad ; \quad (T = 2\pi/\omega) \qquad (9)
$$

2.2.4 Graphical summary of theory

A graphical summary of the basic theoretical results discussed in this chapter is presented in Figures 4 and 5: The results from a stress-strain test ("cigar experiment") on a viscoelastic material are presented in Figure 4. The stress was programmed to follow a sinus variation with time T used per cycle.

Equations 5 and 8 are illustrated by plotting stress applied in the experiment and strain detected respectively against time. Normalized versions of these graphs are shown in Figure 5. Energy loss according to Equation 9 is represented by the shaded area in Figure 4.

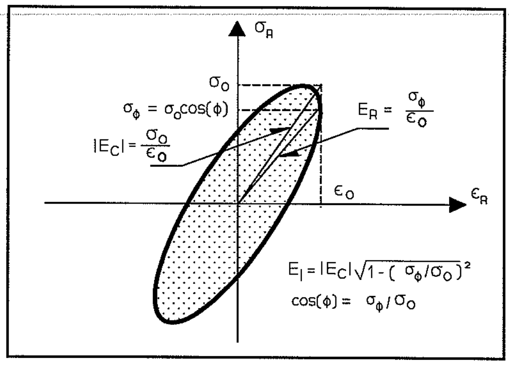

Figure 4. Stress-strain test of viscoelastic material subjected to harmonic vibration. ("Cigar experiment")

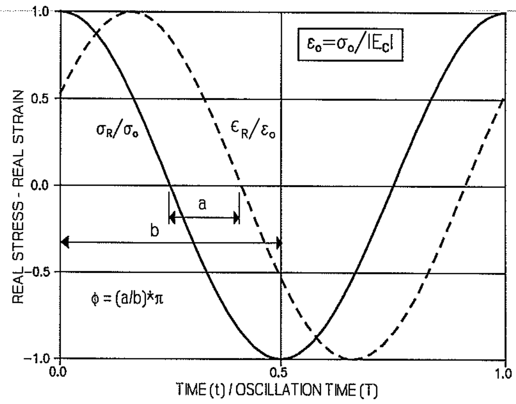

Figure 5. Oscillation time T, ⇒ frequency f = 1/T, (angular frequency ω = 2πf).

Note: *Viscoelastic materials increase their stiffness $|E_C|$ with increasing load frequency. Energy loss is reduced at the same time. The stiffness of elastic materials does not depend on frequency - and there is no energy loss.*

2.2.5 Power law creep

It has often been demonstrated in the literature on creep of materials (ex 13) that many building materials (such as wood) exibit creep which can be fitted sufficiently well by expressions of the power law type $c(t) = (1 + at^b)/E$ where a and b are thought to be material constants. However, the quality of an expression to fit creep data does not necessarily qualify this expression as an appropriate creep function in further viscoelastic studies. It is, for example, very unfortunate that the constant "a" in the power law expression just considered cannot be taken seriously as a real material constant. "a" has the dimension of time raised to minus the other constant "b" involved.

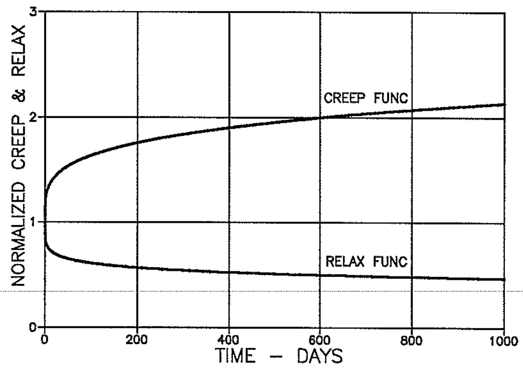

Figure 6. Normalized Power law creep function Ec(t), and normalized relax function r(t)/E (b = 0.25, τ = 600 days).

The power law creep description can, however, very easily be re-formulated to become a qualified creep function in stress-strain analysis of viscoelastic materials. The modified power law creep function presented in Equation 10 is the result of an analysis made in (12,14) on power law creep. The constants now introduced can be considered real material properties. The relaxation time τ (creep doubling time, c(τ) = 2c(0)) decreases with increasing humidity and increasing temperature. The creep power b is usually not influenced very much by climatic conditions.

The relaxation function also presented in Equation 10 was developed in (12) consistently with the concept of power law creep. (Gamma function $\Gamma(1+x) = x!$). An example of associated creep and relaxation is presented graphically in Figure 6.

$$c(t) = \frac{1}{E}\left(1 + \left(\frac{t}{\tau}\right)^b\right) \quad ; \quad \left(\begin{array}{l} b \text{ is creep power (often } b \approx 1/4) \\ \tau \text{ is relaxation time (depends on climate)} \end{array} \right.$$

$$r(t) = E\sum_{k=0}^{\infty} \frac{(-X)^k}{\Gamma(1+kb)} \quad with \quad X = \Gamma(1+b)\left(\frac{t}{\tau}\right)^b \tag{10}$$

$$\approx \frac{1}{c(t)} \quad when \quad b < 0.3$$

Complex stiffness

The complex stiffness associated with Power law creep is presented in Equation 11 reproduced from (12). Special versions are presented in Equation 12 which apply when load oscillates with very high or very low frequencies respectively.

$$|E_C| = \frac{E}{\sqrt{1 + Y^2 + 2Y\cos(b\pi/2)}} \quad ; \quad \tan\phi = \frac{Y\sin(b\pi/2)}{1 + Y\cos(b\pi/2)}$$

$$with \quad Y = \frac{\Gamma(1+b)}{(\tau\omega)^b} = \frac{b!}{(\tau\omega)^b} \tag{11}$$

$$E_R = E\,\frac{1 + Y\cos(b\pi/2)}{1 + Y^2 + 2Y\cos(b\pi/2)} \quad ; \quad E_I = E\,\frac{Y\sin(b\pi/2)}{1 + Y^2 + 2Y\cos(b\pi/2)}$$

$$|E_C| \rightarrow \left(\begin{array}{ll} E & when\ \omega \rightarrow \infty \\ 0 & when\ \omega \rightarrow 0 \end{array} \right. ; \quad \tan(\phi) \rightarrow \left(\begin{array}{ll} 0 & when\ \omega \rightarrow \infty \\ \tan(b\pi/2) & when\ \omega \rightarrow 0 \end{array} \right. \tag{12}$$

Remark: The Power law creep model degenerates with b = 1 to the so-called Maxwell model defined in Appendix A. The stiffness quantities become very simple as demonstrated in Equation 13.

$$|E_C| = E\,\frac{\tau\omega}{\sqrt{1 + (\tau\omega)^2}} = E\cos(\phi) \quad ; \quad \tan(\phi) = \frac{1}{\tau\omega} \quad ; \quad (Maxwell) \tag{13}$$

2.2.6 Example

The viscoelastic behavior of wood can, at certain climatic conditions, be defined by the Power law creep function (Equation 10) with E = 11000 MPa, b = 0.25, τ = 1000 days. The results of a consistent complex stiffness analysis by Equation 11 are shown in Figures 7 and 8.

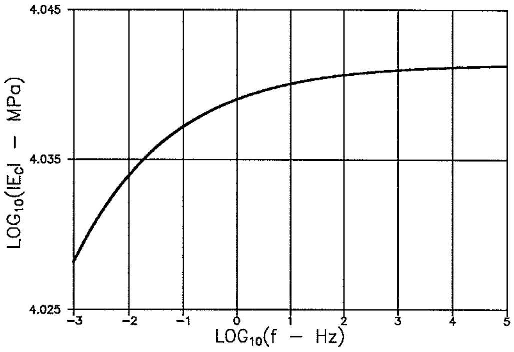

Figure 7. Numerical complex stiffness of wood considered in example, Section 2.2.6.

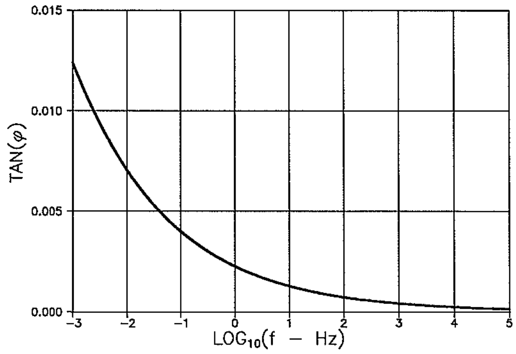

Figure 8. Loss tangent of wood considered in example, Section 2.2.6.

3. Systems

The analysis of viscoelastic systems subjected to harmonic load vibration proceeds along quite similar patterns as used in the analysis of materials. This feature (which is a direct result of the elastic-viscoelastic analogy formulated in Section 2.2.3) is illustrated in this section. The system considered is the bar defined in Figure 9 which can also be thought of as describing the test set-up in an experimental vibration analysis.

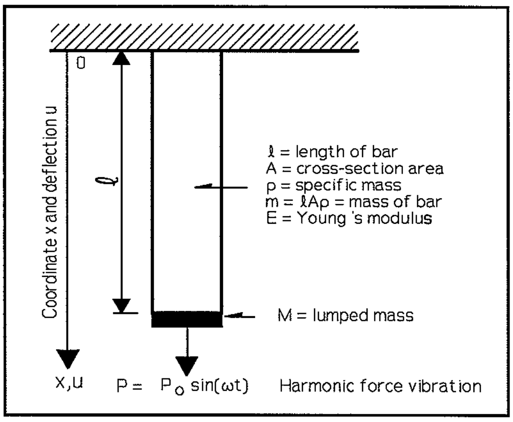

Figure 9. System considered in elastic and viscoelastic analysis.

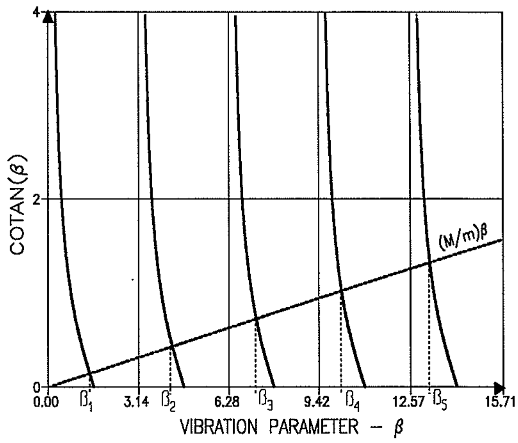

Figure 10. Resonance frequencies. When lumped mass $M = 0$: $\beta_n = (\pi/2)(2n-1)$.

3.1 Elastic system - harmonic vibration

The stiffness (S) of the system considered is expressed by Equation 14 where the so-called vibration parameter is defined in Equation 15 (see Appendix C)

$$S = \frac{P}{u(l)} = \omega^2\left(m\frac{\cot(\beta)}{\beta} - M\right) \qquad (14)$$

$$\beta = \omega\sqrt{\frac{ml}{AE}} \qquad \textit{(elastic vibration parameter)} \qquad (15)$$

It is noticed that stiffness becomes zero at resonance frequencies ω_n determined by Equation 16, see Figure 10.

$$\cot(\beta_n) = \frac{M}{m}\beta_n \Rightarrow \omega_n = \beta_n\sqrt{\frac{AE}{ml}} \qquad \textit{Reson. frequency number n} \qquad (16)$$

3.1.1 Low frequencies

Equation 14 reduces as follows at frequencies well below first resonance frequency where forces of inertia become negligible.

$$S \rightarrow \frac{A}{l} E \quad as \quad \omega \rightarrow 0 \tag{17}$$

3.2 Viscoelastic system - harmonic vibration

We now use the elastic-viscoelastic analogy formulated in Section 2.2.3. This means that the viscoelastic solution to the problem considered is obtained by the elastic solution when stiffness E is replaced by complex stiffness E_C. Thus, the systems complex stiffness is obtained immidiately as follows from Section 3.1.

$$S_C = \omega^2 \left(m \frac{\cot(\beta_C)}{\beta_C} - M \right) \tag{18}$$

$$\beta_C = \omega \sqrt{\frac{ml}{AE_C}} \qquad (complex\ vibration\ parameter) \tag{19}$$

$$\begin{aligned} S_C &= S_R + iS_I = |S_C|\exp(i\delta) && complex\ system\ stiffness \\ |S_C| &= \sqrt{S_R^2 + S_I^2} && numerical\ complex\ system\ stiffness \\ \tan(\delta) &= \frac{S_I}{S_R} && system\ loss\ tangent \end{aligned} \tag{20}$$

3.2.1 Low frequencies

Equation 18 reduces as follows at frequencies well below first resonance frequency where forces of inertia become negligible.

$$S_C \rightarrow \frac{A}{l} E_C \quad as \quad \omega \rightarrow 0 \tag{21}$$

3.2.2 Example

A vibration test is made on a bar made of the wood material previously considered in Section 2.2.6. The wood density is 500 kg/m^3. The test set-up is as shown in Figure 9 with l = 15 cm, A = 9 cm^2, M = 50 gr. The complex stiffness of the bar is calculated by Equation 18. The numerical complex flexibility $|H_C| = 1/|S_C|$ and the loss tangent of the bar are shown in Figures 11 and 12.

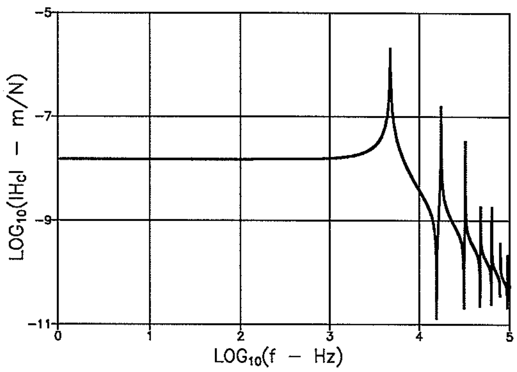

Figure 11. Numerical complex flexibility ($|H_C| = 1/|S_C|$) of system described in example, Section 3.2.2.

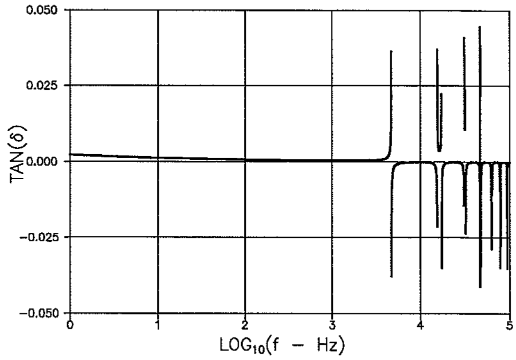

Figure 12. Loss tangent (tan(δ)) of system described in example, Section 3.2.2.

Remark: The data representations chosen in Figures 11 and 12 are those which will appear on a computer screen in modern experimental vibration analysis of systems. Notice that local maxima of flexibilities $|H_C|$ have finite values. The elastic counterparts are infinite.

4. Material versus system

4.1 In general

The results of Section 3.2 can be used as follows to deduce material properties from a system analysis: Solve Equation 18 (the "screen data" in Figures 11 and 12) with respect to β_C. Then, by Equation 19,

$$
\boxed{\begin{aligned}
&E_C = \frac{ml}{A}\left[\frac{\omega}{\beta_C}\right]^2 \;;\; E_R = \Re(E_C) \;;\; E_I = \Im(E_C) \\
&\Rightarrow \tan(\phi) = \frac{E_I}{E_R} \;;\; |E_C| = \sqrt{E_R^2 + E_I^2}
\end{aligned}} \qquad (22)
$$

A convenient algorithm (3,5) for the calculations just outlined is described in Table 1. Newton's principle of iteration is implemented, and works on Y as defined in Equation 23 (based on Equation 18). First β_C-estimate used in this paper is the simple one shown in Table 1. Other estimates can be used. At low frequencies, for example, $\beta_C = \omega(m/S_C)^{0.5}$ is a very accurate estimate (express β_C in Equation 19 with E_C derived from Equation 21).

Table 1. Viscoelastic material stiffness deduced from viscoelastic system stiffness.

```
Given:
Length, cross-section, and mass of bar (l, A, m)
Lumped mass: M
Experimental data: ω, | S_C | = 1/ | H_C | , tan(δ)
Calculations:
     ω               (angular frequency considered)
     β_C = (1 + i*0)  (first estimate, next estim. is β_C from preceeding ω)
4    S_C = | S_C | (cosδ + i sinδ)
1    calculate Y and Y'
     β_C,NEW = β_C - Y/Y'   (Newton's principle of iteration)
     error = | β_C,NEW | / | β_C | - 1
     if ( | error | ≤ 0.0001) go to 2
     β_C = β_C,NY
     go to 1
2    E_C = (ml/A) [ω/β_C]^2
     E_R = ℜ(E_C) ; E_I = ℑ(E_C) ; tanφ = E_I/E_R ; | E_C | = √(E_R² + E_I²)
     write ω, | E_C | , tanφ
     ω = ω + Δω  (ω should vary with same distance on a log-scale)
     go to 4
```

$$
\begin{aligned}
Y &= Y(\beta_C) = m\omega^2\cos(\beta_C) - \beta_C\sin(\beta_C)[M\omega^2 + S_C] = 0 \\
Y' &= \frac{dY}{d\beta_C} = -m\omega^2\sin(\beta_C) - (M\omega^2 + S_C)[\sin(\beta_C) + \beta_C\cos(\beta_C)]
\end{aligned}
\tag{23}
$$

4.2 Special frequencies

Equation 22 reduces considerably when low frequencies and resonance frequencies are approached. The results are subsequently presented in Equation 24, Equation 25 and Figure 13 respectively.

4.2.2 Low frequencies

At low load frequencies (well below first resonance frequency) we can deduce material properties very easily from Equation 21. We get

$$
|E_C| \rightarrow \frac{l}{AE}|S_C| \quad and \quad \phi \rightarrow \delta \quad as \quad \omega \rightarrow 0 \tag{24}
$$

which shows that material properties at low frequencies, where forces of inertia can be ignored, can be read directly from the simple system tests ("cigar experiments") outlined in Figure 4. When bar tests are considered we translate the symbols stress (σ_R), strain (ϵ_R), and stiffnesses (E_R,E_I, $|E_C|$) in Figure 4 to force (P_R), deflection (u_R), and stiffnesses of bar (AE_R/l,AE_I/l,$A|E_C|/l$) respectively.

4.2.1 Resonance frequencies

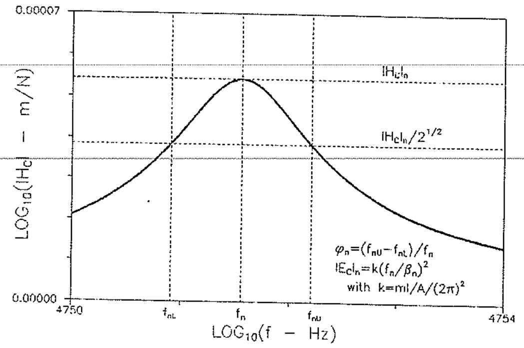

Figure 13. Numerical stiffness and loss angel at resonance frequency n.

It can be shown that complex moduli determined by the method described in Section 4.1 at resonance frequencies (and small loss tangents assumed) equal those predicted by the well-known resonance frequency method (ex 2,15,16). Figure 13 and Equation 25 illustrate how this method "works" on the bar system considered in this paper with vibration parameters β_n determined from Equation 16.

$$
|E_C|_n = \frac{ml}{A}\left[\frac{f_n/(2\pi)}{\beta_n}\right]^2 \; ; \; \phi_n = \frac{f_{nU} - f_{nL}}{f_n} \; ; \quad as\ long\ as\ \phi_n \approx \tan(\phi_n) \tag{25}
$$

Illustration: The example in Section 3.2.2 is recalled. Figure 13 is a blow-up of Figure 11 at the first resonance frequency (n = 1). The following frequencies are observed, (f_{1L}, f_1, f_{1U}) = (4750.91, 4751.56, 4752.21) Hz. Then from Equation 25, $\phi_1 = 0.0002736$, $|E_C|_1 = 11000$ MPa which compare very well with the properties of the bar material described in Section 2.2.6, Figures 7 and 8.

Remark: The resonance frequency method is bound to produce a discrete mapping of complex stiffness versus load frequency - and can be used only on materials with low loss tangents. The method presented in this paper does not have any of these restrictions.

4.3 *Creep and relaxation*

When complex material stiffnes is deduced creep and relaxation properties of the material can be derived by the following expressions presented in (12,14)

$$\boxed{\begin{aligned} r(t) &= E - \frac{2}{\pi}\int_0^{\infty} E_I(\omega)\,\frac{1-\cos(\omega t)}{\omega}\,d\omega \\ c(t) &= \frac{1}{E} + \frac{2}{\pi}\int_0^{\infty} J_I(\omega)\,\frac{1-\cos(\omega t)}{\omega}\,d\omega \quad ; \quad \left(J_I = \frac{E_I}{|E_C|^2}\right) \end{aligned} \qquad (26)}$$

where E is dynamic Young's modulus (= $|E_C|$ at very high frequencies). It is recalled that one of the functions considered can be determined from the other one by Equation 3.

4.3.1 Numerical procedures

Equation 26 is highly theoretical. Complete descriptions of E_I and J_I are assumed from $\omega = 0$ to ∞. Our experimental data are based on a "continuous" distribution of frequencies between a lower value of $\omega = \omega_L$ and an upper value of $\omega = \omega_U$ which means that approximative versions of Equation 26 have to be established in order to predict relaxation functions and creep functions from complex stiffness data available.

First method

Equation 26 can be written as shown in Equation 28 when it is assumed that the integrands vary linearely between $\omega = 0$ and ω_L with a value of 0 at $\omega = 0$. The latter assumption requires $E_I(0)$ and $J_I(0)$ to be 0 or finite quantities (see Equation 27). This is always true for $E_I(0)$. It is true also for $J_I(0)$ when creep functions are considered with *$dc/dt \rightarrow 0$ as $t \rightarrow \infty$ (which applies for the Power law creep function with $b < 1$).*

$$\frac{1 - \cos(\omega t)}{\omega} \rightarrow \frac{1}{2}\omega t^2 \quad for\ \omega \rightarrow 0 \tag{27}$$

It is further more assumed in Equation 28 that ω_U is high enough to simulate very high frequencies ($E_{DYN} \approx |E_C(\omega_U|$).

$$\begin{aligned} r(t) &= |E_C(\omega_U)| - \frac{2}{\pi}\left[\frac{1-\cos(\omega_L t)}{2}E_I(\omega_L) + \int_{\omega_1}^{\omega_U} E_I(\omega)(1-\cos(\omega t))\,d\log(\omega)\right] \\ c(t) &= \frac{1}{|E_C(\omega_U)|} + \frac{2}{\pi}\left[\frac{1-\cos(\omega_L t)}{2}J_I(\omega_L) + \int_{\omega_1}^{\omega_U} J_I(\omega)(1-\cos(\omega t))\,d\log(\omega)\right] \end{aligned} \tag{28}$$

General method

From preceding section is known that relaxation functions can always be determined by Equation 28. Creep functions, however, can only be determined by this expression when dc/dt $\rightarrow$ 0 as t $\rightarrow \infty$. If this condition is violated (or it is unknown) the latter expression in Equation 3 can be used numerically to convert a known relaxation function to its associated creep function. A very rough algorithm for this purpose is shown in Equation 29. More refined algorithms are found in (4,17). Before using the algorithms a continuous r(t) description must be established from the discrete r(t)-quantities determined by Equation 28 (computer-fit with r(0) = E_{DYN}).

$$\begin{aligned} \Delta c(t_1) &= \frac{1}{E_{DYN}} \quad ; \quad t_1 = 0 \\ \Delta c(t_N) &= \frac{1}{E_{DYN}}\left[1 - \sum_{n=1}^{N-1} \Delta c(t_n) r(t_N - t_n)\right] \quad ; \quad N = 2, 3, 4, \ldots \\ c(t_N) &= \sum_{n=1}^{N} \Delta c(t_n) \end{aligned} \tag{29}$$

4.3.2 Other rheological properties

Creep functions or relaxation functions are the principal material properties of viscoelastic materials in the area of building materials. In other areas alternate properties may very well be more appropriate. Examples are relaxation spectra and retardation spectra in polymer science. Various information on theoretical and experimental coupling between rheological characteristics of materials can be found in (ex 12,18,19,20).

5. Example

The following "experimental" vibration analysis has been made to test the procedures presented in chapter 4: The wood material previously considered in Sections 2.2.6 and 3.2.2 has been made wet such that density, Young's modulus (dynamic stiffness), and relaxation time are now 700 kg/m^3, E = 9000 MPa, and τ = 30 days. Creep power is maintained at b = 0.25. (We know from other sources about these changes of properties. The material, however, does not).

The test set-up is maintained at l = 15 cm, A = 9 cm^2, M = 50 gr. The "experimental" data (based on Equation 18) shown in Figures 14 and 15 appear on the screen. It is assumed that signals in the experimental vibration analysis are emitted and received without any noise or numerical disturbances.

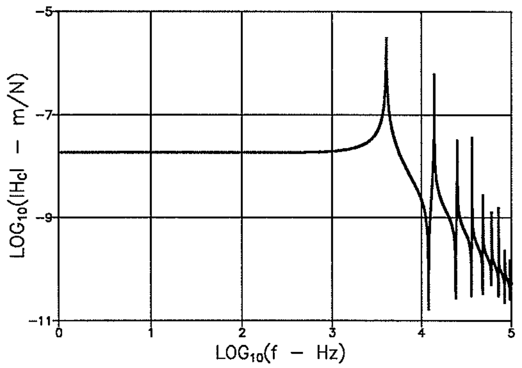

Figure 14. "Experimental" data in example, Chapter 5. Numerical complex flexibility of system tested.

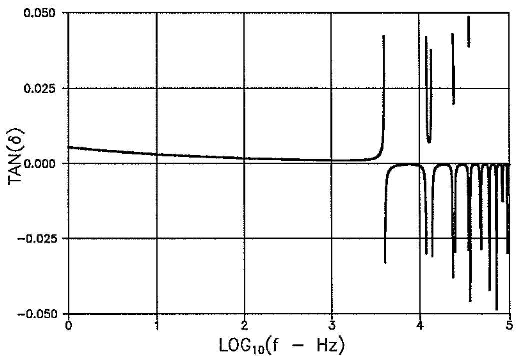

Figure 15. "Experimental" data in example, Chapter 5. Loss tangent of system tested.

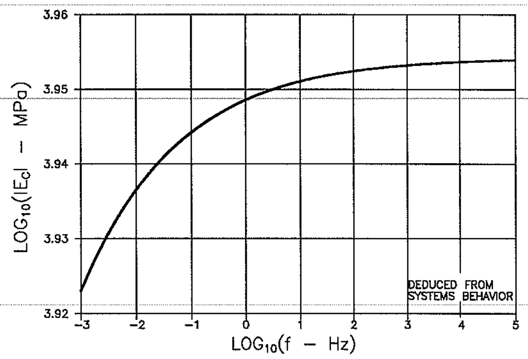

Figure 16. Numerical complex material stiffness deduced from test on system described in example, Chapter 5.

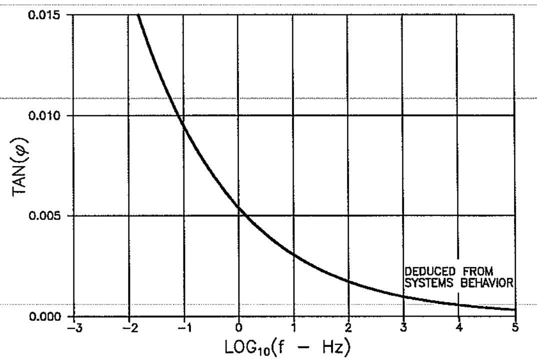

Figure 17. Loss tangent of material deduced from test on system described in example, Chapter 5.

The algorithm in Table 1 is now used on the "experimental" data. The resulting viscoelastic properties of the material tested are shown in Figures 16 and 17.

Equation 28 is then used on these data to determine creep and relaxation of the material (for simplicity of the example: we know from other sources that dc/dt never stops decreasing). The results are shown by dots in Figures 18 and 19.

Remark: It has previously been mentioned that qualified estimates on 0- and ∞-quantities are required in the numerical handling of Equation 28. These estimates depend on lower frequency f_L and upper frequency f_U applied when collecting the input-data (Figures 14 and 15) by experimental vibration analysis. The Brüel & Kjær equipment (Type 2035) used at the Building Materials Laboratory, Technical University of Denmark has an upper limit of $f = 10^5$ Hz which is therefor used as a fixed f_U. The lower frequencies of $f_L = 10^{-3}$ Hz and 10^{-5} Hz used to obtain the r(t) and c(t) results presented in Figures 18 and 19 have the purpose of testing the sensitivity of the numerical procedure applied with respect to estimated 0-quantities. The subdivision of frequencies applied in this example was 5000 on a log-scale between f_L and f_U.

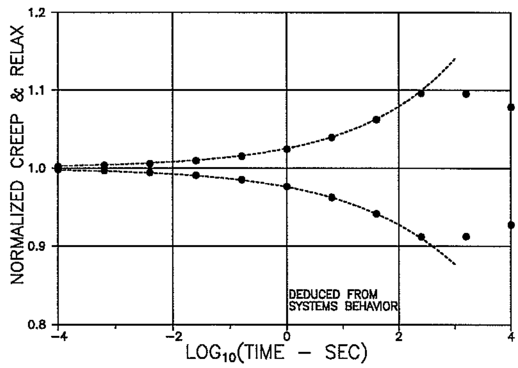

Figure 18. c(t) and r(t) from system tests. Test freq. $(f_L, f_U) = (10^{-3}, 10^5)$ Hz. $|E_C|$ at f_U used as E (E_{DYN}).

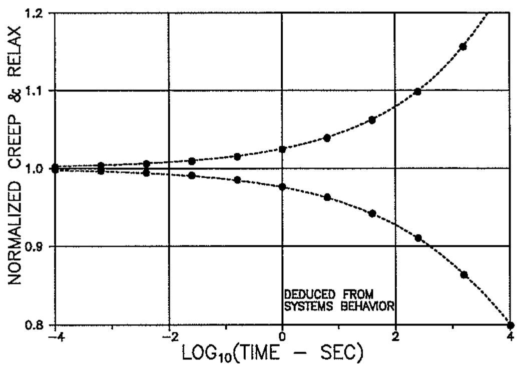

Figure 19. c(t) and r(t) from system tests. Test freq. $(f_L, f_U) = (10^{-5}, 10^5)$ Hz. $|E_C|$ at f_U used as E (E_{DYN}).

5.1 Discussion

The procedures presented in this chapter on how to deduce material properties from vibration tests are clearly justified by the example considered. We know from Equation 10 the true answers to the creep- and and relaxation functions of the material considered. They are shown in Figures 18 and 19 by dotted lines.

The figures indicate that $f_U = 10^5$ Hz is adequate as an upper limit in the integration process of Equation 28. The question of a qualified lower frequency for integration is answered as would be expected: To get creep and relaxation right at longer times t we must use a lower frequency of magnitude $f_L \leq \approx 0.1/t$. This means that supplementary tests of the kind discussed in Sections 2.2.4 and 4.2.2 and illustrated in Figure 4 become relevant when creep/relaxation properties for t > a few minutes are of interest. The significance of subdivision of frequency range has not been considered. The Brüel & Kjær equipment previously referred to has a frequency subdivison of 800 on a linear scale (can be changed to log-scale).

It is emphasized that only the plain mathematical procedures explained in Section 4 are evaluated in this example. No attemps have been made to evaluate the influence on the results obtained of noise and numerical disturbances associated with transfer of signals in practice.

6. Perspectives and final remarks

The method explained in this paper on material properties determined by experimental vibration analysis of systems has been demonstrated on a bar system. The method, however, applies to any test system for which the elastic vibration solution is known. Examples of appropriate test specimens for which such solutions are available in an analytical form are simple beams or plates with different boundary conditions (ex 15,21,22,23). An example of the method applied to a fixed-free test beam in bending is presented in (24). Recent studies (25) have been made on how to couple the method to numerically determined elastic solutions (FEM).

When the method presented is used on data obtained from modern experimental vibration analysis of systems (using B&K type 2035 for example) rheological information of materials can, theoretically, be detected in the range of time, from 10^{-5} seconds to a couple of minutes. This range (in practice: from a very small fraction of a second to a couple of minutes) is very important from a materials science point of view. Basic mechanisms can be detected or estimated which control the materials long-term behavior such as creep, relaxation, and damping (loss tangent) for example.

Examples from practice where such information can prove very valuable in the area of building materials are: Quality control in materials production, non-destructive testing of materials such as concrete and wood, quantification of progressing materials decomposition due to freezing or salt infection, change of materials behavior with respect to temperature such as in cement- and asphaltic concretes, change of composite materials behavior in general with respect to mixture variations.

Theoretically there is no problem in increasing the working range of an experimental vibration analysis from the few minutes previously mentioned to longer times. For long time studies, however, it is more appropriate to combine experimental vibration analysis (< 1 minute) with alternate methods such as the "cigar" experiment outlined in Figure 4 (< 1 day) - and ordinary creep experiments (> 1 day). As creep functions and relaxation functions are related through Equation 3 the latter method can be replaced by ordinary relaxation experiments.

Appendix A

Complex numbers

Calculation with complex numbers is an essential mathematical element in any vibration analysis. Some useful expressions reproduced from the mathematical literature (ex 26) are presented in this section:

Complex number A_C. Real part A_R. Imaginary part A_I. Numerical or absolute value $|A_C|$. Delay angle θ. Imaginary unit is i.

Algebraic notation:

$$A_C = A_C(i) = A_R + i\,A_I \quad ; \quad \tan\theta = A_I / A_R \qquad \text{(A1)}$$
$$|A_C| = \sqrt{A_R^2 + A_I^2} = \sqrt{A_C(i) * A_C(-i)}$$

Trigonometric notation:

$$A_C = |A_C|[\cos\theta + i\,\sin\theta] \quad ; \quad \tan\theta = A_I / A_R \qquad \text{(A2)}$$
$$A_R = |A_C|\cos\theta \quad ; \quad A_I = |A_C|\sin\theta$$

Exponential notation:

$$A_C = |A_C|\exp(i\theta) = |A_C|[\cos\theta + i\,\sin\theta] \qquad \text{(A3)}$$
$$i = \exp\left(i\frac{\pi}{2}\right) \quad ; \quad i^b = \exp\left(i\frac{b\pi}{2}\right) = \cos\left(b\frac{\pi}{2}\right) + i\,\sin\left(b\frac{\pi}{2}\right)$$

Others:

$$|A_C|^2 = A_C(i) * A_C(-i)$$
$$A_R = \frac{1}{2}[A_C(i) + A_C(-i)] \quad ; \quad A_I = -\frac{i}{2}[A_C(i) - A_C(-i)]$$
$$\tan\theta = -i\,\frac{A_C(i) - A_C(-i)}{A_C(i) + A_C(-i)} \qquad \text{(A4)}$$
$$A_C^{-1} = \frac{1}{A_C} = \frac{\exp(-i\theta)}{|A_C|} = \frac{A_C(-i)}{|A_C|^2} = \frac{A_R}{|A_C|^2} - i\,\frac{A_I}{|A_C|^2}$$
$$A_C^n = |A_C|^n \exp(in\theta) = |A_C|^n [\cos(n\theta) + i\,\sin(n\theta)]$$

Appendix B

Some basic viscoelastic models

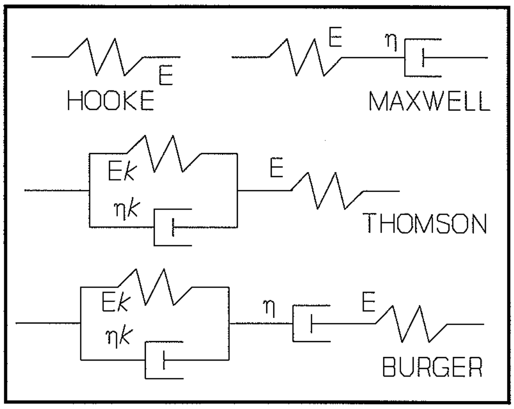

Figure B1. Viscoelastic material models. Appropriate combinations of material parameters are presented in Equation B1.

A summary based on (11,12,14) is presented in this appendix on the rheology of some basic viscoelastic materials. Creep functions c(t), relaxation functions r(t), and complex stiffnesses E_C of the models shown in Figure B1 are considered. The latter quantities are presented in a very condensed form which needs Appendix A to be split up into real stiffness E_R and imaginary stiffness E_I.

Abreviations

$$\tau = \frac{\eta}{E} \quad ; \quad \tau_K = \frac{\eta_K}{E_K} \quad ; \quad \alpha = \frac{E}{E_K} \quad ; \quad m_T = 1 + \alpha$$

$$\begin{matrix} m_{B1} \\ m_{B2} \end{matrix} = \frac{1}{2}\left(1 + \alpha + \frac{\tau_K}{\tau} \pm \sqrt{\left(1 + \alpha + \frac{\tau_K}{\tau}\right)^2 - 4\frac{\tau_K}{\tau}}\right) \tag{B1}$$

Creep and relaxation

HOOKE: $c(t) = \dfrac{1}{E} \quad ; \quad r(t) = E$

MAXWELL: $c(t) = \dfrac{1}{E}(1 + t/\tau) \quad ; \quad r(t) = E\exp(-t/\tau)$

THOMSON: $c(t) = \dfrac{1}{E}\left[1 + \alpha\left(1 - \exp(-t/\tau_K)\right)\right]$

$$r(t) = \frac{E}{1 + \alpha}\left[1 + \alpha\exp(-m_T t/\tau_K)\right] \tag{B2}$$

BURGER: $c(t) = \dfrac{1}{E}\left[1 + \dfrac{t}{\tau} + \alpha\left(1 - \exp(-t/\tau_K)\right)\right]$

$$r(t) = \frac{E}{m_{B1} - m_{B2}}\left[(m_{B1} - 1)\exp(-m_{B1}t/\tau_K) - (m_{B2} - 1)\exp(-m_{B2}t/\tau_K)\right]$$

Complex stiffness

The complex stiffness is obtained from the so-called analogue stiffness $E^A = E^A(s)$ by replacing the argument s with iω, meaning $E_C = E^A(i\omega)$.

$$\begin{array}{ll} \textit{HOOKE}: & E^{A} \equiv E \\ \textit{MAXWELL}: & E^{A} = E\dfrac{\tau s}{1 + \tau s} \\ \textit{THOMSON}: & E^{A} = E\dfrac{1 + \tau_K s}{m_T + \tau_K s} \\ \textit{BURGER}: & E^{A} = E\dfrac{\tau_K s\ (1 + \tau_K s)}{(m_{B1} + \tau_K s)(m_{B2} + \tau_K s)} \end{array} \tag{B3}$$

Power Law creep: A material with Power Law creep, defined in Equation 10, has the following analogue stiffness

$$\textit{POWER LAW}: \quad E^{A} = E\frac{(\tau s)^b}{\Gamma(1 + b) + (\tau s)^b} \tag{B4}$$

$$\textit{with } \Gamma\textit{-function } \Gamma(1 + b) = b!$$

Appendix C

Elastic system analysis

The elastic force-deflection relation of the bar system considered in the main text of the paper (se Figure 9) is developed as follows from the mechanical model outlined in Figure C1 with forces of gravity ignored. The solution can also be reproduced directly from the literature (ex 23,22).

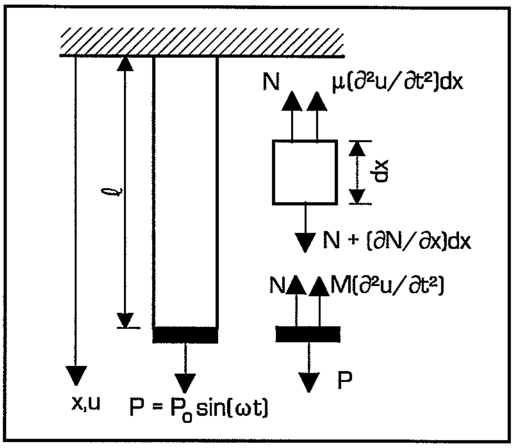

Figure C1. System considered. Mass per length unit of bar is μ = $A\rho$ where ρ is density of material. M is lumped mass.

Geometry, equilibrium, and physical condition (Hooke's law) requires that the bar element shown in Figure C1 will move (u) with position (x) and time (t) as described in Equation C1.

$$\left.\begin{aligned} \frac{\delta N}{\delta x} &= \mu\frac{\delta^2 u}{\delta t^2} \quad \textit{Equilibrium} \\ \frac{\delta u}{\delta x} &= \frac{N}{AE} \quad \textit{Hooke's law} \end{aligned}\right\} \Rightarrow \frac{\delta^2 u}{\delta t^2} = \frac{AE}{\mu}\frac{\delta^2 u}{\delta x^2} \quad \textit{Basic deflection} \qquad \text{(C1)}$$

which can be split up into two differential equations as shown in Equation C2. The influence of time on deflection is considered in a time function T. The influence of position on deflection is considered in a position function U. The two functions are coupled through the constant $\underline{\omega}$ the significance of which as a frequency will appear in the subsequent analysis.

$$\begin{aligned} &u = U(x)T(t) \Rightarrow \\ &\frac{1}{T}\frac{d^2T}{dt^2} = \frac{AE}{\mu}\frac{1}{U}\frac{d^2U}{dx^2} = \textit{constant} = \underline{-\omega^2} \end{aligned} \qquad \text{(C2)}$$

The two equations are solved in Equation C3 with constants A_1, A_2, (or A_3, γ), B_1, and B_2 to be determined from the boundary conditions shown in Equation C4.

$$\frac{d^2T}{dt^2} + \underline{\omega}^2 T = 0 \;\Rightarrow\; T = A_1\cos(\underline{\omega}t) + A_2\sin(\underline{\omega}t) = A_3\,\sin(\underline{\omega}t + \gamma)$$

$$\frac{d^2U}{dx^2} + \left(\frac{\beta}{l}\right)^2 U = 0 \;\Rightarrow\; U = B_1\cos\left(\beta\frac{x}{l}\right) + B_2\sin\left(\beta\frac{x}{l}\right) \quad \textit{with} \qquad \text{(C3)}$$

$$\textit{frequency parameter} \quad \underline{\beta} = \underline{\omega}\sqrt{\frac{ml}{AE}} \qquad (m = Al\mu \;\; \textit{is mass of bar})$$

The boundary conditions are at any time, see Figure C1,

$$\begin{aligned} &at\ x = 0: \quad u(0) \equiv 0 \\ &at\ x = l: \quad N = AE\frac{\delta u}{\delta x} \equiv P_o\sin(\omega t) - M\frac{\delta^2 u}{\delta t^2} \end{aligned} \qquad \text{(C4)}$$

by which the general solution expressed by Equation C2 with U and T from Equation C3 is reduced as follows ($\underline{\omega}$ = load frequency ω, $B_1 = 0$, and $\gamma = 0$).

$$u(x) = \frac{P_0\sin(\omega t)\sin(\beta x/l)}{\omega^2[(m/\beta)\cos\beta - M\sin\beta]} \qquad (\textit{System deflection})$$

$$\textit{with} \quad \beta = \omega\sqrt{\frac{ml}{AE}} \qquad (\textit{vibration parameter}) \qquad \text{(C5)}$$

with *resonance frequencies* (frequencies ω_n at which $u \to \infty$) determined by

$$\cot(\beta_n) = \frac{M}{m}\beta_n \;\Rightarrow\; \omega_n = \beta_n\sqrt{\frac{AE}{ml}} \qquad \textit{Reson-freq. no } n = 1,2,3,..$$

$$\omega_n = \frac{\pi(2n-1)}{2}\sqrt{\frac{AE}{ml}} \qquad \textit{when} \quad M = 0 \qquad \text{(C6)}$$

Literature

1. Døssing, O. (Brüel & Kjær) : "Structural testing - Part I: Mechanical mobility measurements" and ".... - Part II: Modal analysis and simulation", Brüel & Kjær, Nærum, Denmark, 1988.

2. Anonymous (Brüel & Kjær): "Complex modulus apparatus: Instructions and applications", Brüel & Kjær, Nærum Denmark, 1968.

3. Nielsen, L. Fuglsang: "Bjælkesvingninger: Elastiske og viskoelastiske, Bestemmelse af $E_C(\omega)$ ved dæmpede frie svingninger, harmonisk response og resonansfrekvens", (in danish, Vibration of elastic and viscoelastic beams: Complex Young's modulus derived from damped free vibration, harmonic response, and resonance frequency), Research- and text note, Build. Mat. Lab., Tech. Univ. Denmark, 1985. Reproduced and used in ref. 5 (L. Nøhr Larsen).

4. *Idem:* "Materialemekanik", (in danish, Material Mechanics), Textbook in Course 6110: Material Mechanics, Build. Mat. Lab., Tech. Univ. Denmark, Tech. Report 189(1988).

5. Larsen, L. Nøhr: "Træs dynamiske egenskaber - specielt ved lave frekvenser", (in danish, Dynamic properties of wood - especially at low frequencies). M.Sc. thesis, Build. Mat. Lab., Tech, Univ., Denmark, 1985.

6. Flügge, W.: "Viscoelasticity", Blaisdell Publ. Comp., London 1967

7. McHenry, D.: "A new aspect of creep in concrete and its application to design", ASTM Proc., 43(1943), pp 1069-1084.

8. *Idem:* "Toward a generalized treatment of delayed elasticity in concrete", In Internationale Vereinigung für Brückenbau und Hochbau, Abhandlungen, Band 26(1966), pp 269-283, Eidgenössische Technische Hochschule, Zürich.

9. Zienkiewicz, O.C.: "Analysis of viscoelastic behavior of concrete with particular reference to thermal stresses", Proc. Am. Concr. Inst., 58(1961), pp 383-394.

10. Nielsen, L. Fuglsang: "Effects of Creep in Uncracked Composite Structures of Steel and Concrete", Bygningsstatiske Meddelelser (Denmark), 38(1967), 65.

11. *Idem:* "Rheologische Eigenschaften für isotrope linear-viskoelastische Kompositmaterialien". Cement & Concrete Res., 3(1973), 751 - 766.

12. *Idem:* "Power Law Creep as Related to Relaxation, Elasticity, Damping, Rheological Spectra and Creep Recovery - With Special Reference to Wood", Conference of Int. Union of Forestry Research Organizations (IUFRO), Xalapa, Mexico, December 1984. Proc., Build. Mat. Lab., Techn. Univ. Denmark 1985, 181 - 204.

13. Nielsen, A.: "Rheology of building materials", thesis, Statens Institut för Byggnadsforskning, Stockholm, Document D6(1972).

14. Nielsen, L. Fuglsang: "Power-Law creep of wood - composite and dynamic aspects", Conference of Int. Union of Forestry Research Organizations (IUFRO),

Bordeaux - France, 17.- 21. august, 1992. Proc., Build. Mat. Lab., Techn. Univ. Denmark, 1992, 25 - 33.

15. Hansen, P. Freiesleben: "Resonansfrekvensmåling" (in dansish, Resonance frequency measurements), Materials Section DIA, Tech. Univ. Denmark, 1968.

16. Trudsø, E.: "Lydtekniske målemetoder, anvendt på byggematerialer", (in danish, Sound measurements on building materials), Ingeniøren (Copenhagen), no. 2(1958), 73-80.

17. Madsen, K.: "Matrix formalism for practical solutions of concrete creep problems", DIALOG 1-79, Structural Division, Danmarks Ingeniørakademi, Tech. Univ. Denmark, 1979.

18. Nowick, A.S., and Berry, B.S.: "Anelastic relaxation in crystalline solids", Academic Press, New York, 1972.

19. Leaderman, H.: "Viscoelastic phenomena in amorphous high polymeric systems", Chapter 1 in Eirich, F. (ed): "Rheology", Vol. II, Academic Press Inc., New York, 1958.

20. Schwarzl, F.R.: "Numerical calculation of stress relaxation modulus from dynamic data for linear viscoelastic materials", Rheologica Acta, 14(1975), pp 581-590.

21. Pickett, G.: "Equations for computing elastic constants from flexural and torsional resonant frequencies of vibration of prisms and cylinders", Proc. ASTM, VOL 45(1945), 846-865.

22. Nowacki, W.: "Dynamics of elastic systems", Chapman & Hall Ltd., London, 1963.

23. Timoshenko, S.: "Vibration problems in engineering", 2d ed., D. Van Nostrand Co., New York, N.Y., pp. 337-342, 1937.

24. Nielsen, L. Fuglsang: "Viskoelastiske materialeegenskaber versus målte viskoelastiske bjælkeegenskaber", (in danish, Viscoelastic material properties versus measured viscoelastic beam properties), Build. Mat. Lab., Tech. Univ. Denmark, Tech. report 292(1993).

25. Vestergaard, B.: "Identifikation af modeller for kompleks elasticitetsmodul af viskoelastiske materialer", (in danish, Identification of models for complex elastic moduli of viscoelastic materials), Ph.D. Thesis, Dept. Solid Physics, Tech. Univ. Denmark, DCAMM report S58(1991).

26. Bronstein, I.N. and Semendjajew, K.A.: "Taschenbuch der Mathematik", 4. Auflage, Verlag Harri Deutsch, Frankfurt/M. u. Zürich, 1964.