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State of the Art of Numerical Simulation and Computational Models in Coupled Instabilities

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STATE OF THE ART OF NUMERICAL SIMULATION AND COMPUTATIONAL MODELS IN COUPLED INSTABILITIES

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Abstract

The paper provides a report on the simulation and computational models of structures prone to interaction buckling. The simulation models are divided into perturbation, cell and discrete models, and all major contribution to the research of interaction buckling are classified using this division. The paper also explores current trends in the development of simulation models and likely future research areas.

1. Introduction

Since the 1950s, nautical and aeronautical engineers as well as building and bridge engineers have shown growing interest in thin-walled structures. The interest has been driven by industry demand for light weight structural components, enhanced structural efficiency and reduced material use.

The fact that thin-walled structures are prone to buckling in several modes posed a challenging nonlinear problem which attracted theoreticians. The phenomena characterising interaction buckling, including imperfection sensitivity, and change and localisation of buckling patterns, were clarified during the 1960s and 1970s mainly by using simple simulation models. At the same time, thin-walled structural components became increasing popular in practice and demands for more efficient design rules increased.

By the mid 1970s, a gap between theory and practice became apparent. The simple simulation models, which were based largely on perturbation methods, were useful from the viewpoint of explaining the phenomena of interaction buckling. However, the methods were too simplistic to provide design data, and consequently, the development of design rules was empirically based. The gap between theory and practice was narrowed during the 1980s and early 1990s as a result of developments in computing technology. These developments nurtured research in computational methods and spawned the finite strip and finite element methods.

The schools of computational methods formed alliances with both theoreticians and practicians. As a result, it is now becoming standard practice to use numerical methods in conjunction with experiments in refining and formulating new design rules for thin-walled structures. Likewise, perturbation methods have been combined with advanced numerical methods to provide simple mechanical models of complex structures. These models play an important role in explaining fundamental behaviour.

The purpose of this paper is to summarise developments in the simulation and computation of thin-walled structures and to give an indication of current trends and likely future research.

2. Simulation models

2.1 Classification

The theoretical research of thin-walled structures can be divided into three main groups - here termed "simulation models". These are the perturbation models, the cell models and the discrete models, as summarised in Table 1. The perturbation models seek asymptotically exact solutions and usually aim at simple models for explaining fundamental behaviour. The cell models represent an engineering approach to interaction buckling by using the *stiffness* of the locally buckled cross-section in an overall member analysis. The discrete models involve a full discretisation of the thin-walled member.

The simulation models can be subdivided into "computational models" for further classification. The most common computational models are the semi-analytical and spline finite strip methods, and the finite element method, as shown in Table 1. The computational models can again be subdivided into "types of analysis". These are summarised in Table 1 as linear, bifurcation (or buckling), nonlinear bifurcation, geometric nonlinear, material nonlinear, and geometric and material nonlinear analyses.

Simulation models	Computational models	Types of analysis
Perturbation	Analytical	Linear
Cell	Semi-analytical finite strip	Bifurcation
Discrete	Spline finite strip	Nonlinear bifurcation
	Finite element	Geometric nonlinear
	Finite difference	Material nonlinear
		Geometric and material nonlinear

Table 1: Simulation models, computational models and types of analysis.

2.2 Review of simulation models

The main contributions to the development of simulation models are described in this section. The review does not purport to be a complete summary of available literature on simulation and computational models. Rather, it describes the sequence of major developments and provides reference to key papers enabling a more thorough review of particular models.

Perturbation model. Arguably, this model proved the most important and successful model in the early research of interaction buckling. Koiter [65] provided the foundation of the model by applying the perturbation technique to structural mechanics. Important extensions of Koiter's theory included the analyses of dynamic [20] and inelastic [61] buckling as well as the analysis of structures with non-simultaneous buckling loads [25].

Using the formulation of [25], the perturbation model can be summarised as follows: The displacements, strains and stresses are expanded in perturbation series,

$$u = \lambda u_0 + \xi_i\, u_i + \xi_i \xi_j\, u_{ij} \ldots \tag{1}$$

$$\varepsilon = \lambda \varepsilon_0 + \xi_i\, \varepsilon_i + \xi_i \xi_j\, \varepsilon_{ij} \ldots \tag{2}$$

$$\sigma = \lambda \sigma_0 + \xi_i\, \sigma_i + \xi_i \xi_j\, \sigma_{ij} \ldots \tag{3}$$

in which summation is implied on repeated indices and λ is a load factor. In eqns. (1-3), u, ε and σ are vectors of dispacements, strains and stresses, and ξ_i, $i = 1, 2, \ldots, M$, are perturbation parameters associated with M buckling modes, u_i, $i = 1, 2, \ldots, M$. The subscript 0 refers to the prebuckling field. A single Latin subscript refers to the primary field (consisting of buckling modes), a double Latin subscript refers to the secondary field, etc. By confining the analysis to fields up to second order, the equilibrium equations may be written in the form,

$$\xi_I\,(1 - \lambda/\lambda_I) + \xi_i \xi_j\, a_{ijI} + \xi_i \xi_j \xi_k\, b_{ijkI} = \lambda/\lambda_I\, \bar{\xi}_I, \qquad I = 1, 2, \ldots, M \tag{4}$$

where λ_I is the eigenvalue of the Ith buckling mode and $\bar{\xi}_I$ is the imperfection component in the shape of the Ith buckling mode. The a_{ijk} and b_{ijkl} coefficients are functions of the design parameters of the structure and are determined such that the a_{ijk} coefficient depends solely on the first order field, whereas the b_{ijkl} coefficient depends on both the first and second order fields.

The perturbation method proved successful because firstly, the displacement field (u) could be obtained as a sum of simple solutions of a linear eigenvalue problem and a set of linear inhomogeneous equations, and secondly, the equilibrium equations entailed only as many degrees of freedom as the number of participating buckling modes. Today, it is of less concern that the displacement field can be obtained from linear problems. However, the ability to produce simple mechanical models by virtue of yielding equilibrium equations in a few variables (ξ_i) remains an attractive feature of the perturbation model.

Several methods have been devised for determining the displacement fields $(u_i, u_{ij}, \ldots)$. Early applications concerned simple structures, including the van der Neut column [22, 67], the truss column [21, 111], the infinitely wide stiffened panel [68, 113], and stiffened axisymmetric cylinders [25, 66], for which the displacement fields, and hence the coefficients of the equilibrium equations, could be obtained analytically. Aiming for solutions for general cross-sectional shapes of prismatic members, the semi analytical finite strip method was used [11, 12, 45, 88, 94, 108] to determine the 1st and 2nd order fields of the asymptotic expansion. The method was based on harmonic and polynomial shape functions for the longitudinal and transverse directions respectively. Consequently, the applications were limited to structures loaded at the ends through boundary conditions which could be satisfied by harmonic functions.

More recently, the spline finite strip method [87, 115] has been combined with a perturbation method. This combination is particularly useful for analysing thin-walled members with arbitrary end support conditions and members undergoing localised distortional deformations as a result of concentrated transverse loads. The finite element method has also been applied to determine the first and second order fields of the asymptotic expansion [23, 24, 49, 71].

It is well known that numerical difficulties may arise when determining second order fields. The difficulties occur because finite elements may lead to membrane locking and because the second order fields must be orthogonal to the first order fields. The difficulties are usually overcome by implementing constraints in the field equations by means of Lagrange multipliers [23, 88, 115].

The aforementioned applications of the perturbation model have assumed linear-elastic materials. Hutchinson [60, 61] extended the asymptotic theory to inelastic material behaviour. However, the complexities associated with accounting for elastic unloading from the yield surface in applying this theory are considerable and only few applications [114] involving the interaction of buckling modes have been reported.

<u>Cell model.</u> This model is particularly suited to members buckling interactively in local and overall modes. Local buckling is here defined as a mode of deformation which does not involve deflections at plate junctions [13] such that the nonlinear von Karman plate equations apply. The underlying idea of the cell model is to construe the structure as a one-dimensional member (or beam). This allows the effect of local buckling to be considered simply by modifying the stress-strain relations of the member. The stress-strain relations of the locally buckled member can be obtained from a separate analysis of a length of section equal to the local buckle half-wavelength (a cell), as shown in Fig. 1. The procedure assumes that the local buckle half-wavelength is an order of magnitude shorter than that of the overall mode.

For columns bifurcating in an overall flexural mode, the analysis simplifies to determining the flexural rigidity of the locally buckled member and using the reduced rigidity in the Euler buckling formula. This type of analysis is described as a "nonlinear bifur-

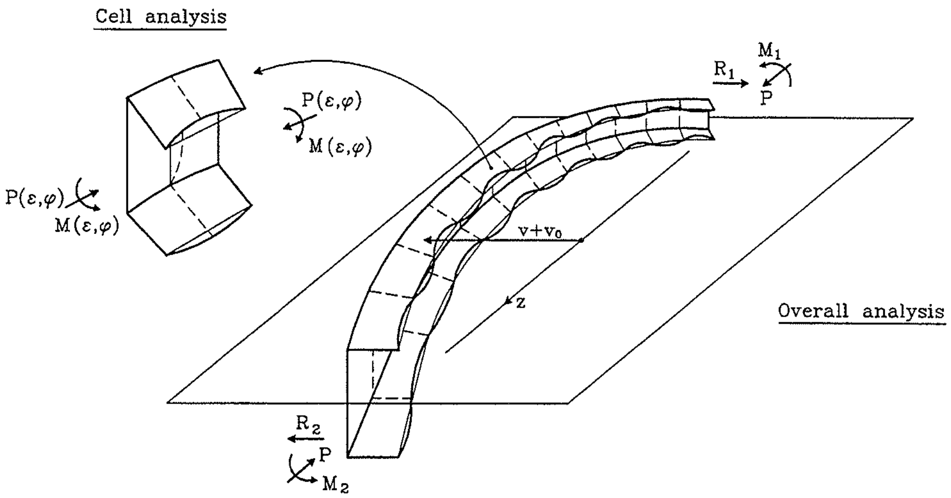

Figure 1: Cell model.

cation analysis", since the pre-bifurcation state is nonlinear. The rigidity of the locally buckled member is obtained by subjecting a cell to a small curvature, calculating the resulting bending moment and determining the rigidity as the ratio of moment to curvature. Several methods have been devised to determine the flexural rigidity of locally buckled members, including split rigidities [14], analytic solutions [46], effective widths [38, 63] and finite strips [52].

The cell model has also been applied to the lateral-torsional bifurcation of locally buckled beams [18, 117, 118]. The model was formally described in [99] and applied to analyse the bifurcation of locally buckled singly symmetric columns in flexural and flexural-torsional overall modes.

To include the effect of overall geometric imperfections (v_0), the cell model was extended to a geometric nonlinear in-plane beam-column analysis [35, 54, 100] by combining the finite difference method described in [28, 50] with nonlinear stress-strain relations, as obtained from an elastic finite strip analysis [53] of a locally buckled cell. The stress-strain relations were obtained *prior* to the beam-column analysis by subjecting a cell to a large number of combinations of axial strain (ε) and curvature (ϕ). The results of this analysis were stored on disk in arrays representing axial force (P) and bending moment (M) as functions of axial strain and curvature. The beam-column analysis was based on a matrix formulation [28, 50] of Newmark's method [90] using a Newton-Raphson iteration scheme. Thus, the equilibrium equation, expressed in the deflection (v),

$$v(z) \quad = \quad \frac{1}{P}\left(M(\varepsilon, \phi) - M_1 - R_1\, z - P\, v_0(z)\right) \tag{5}$$

$$\begin{aligned}
\phi(z) &= -v''(z) && (6)\\
\varepsilon(z) &= \varepsilon(P, \phi(z)) && (7)
\end{aligned}$$

was solved numerically using a finite difference approach.

The cell model is highly efficient in the case of elastic interaction, for which the stress-strain relations can be obtained prior to the beam-column analysis. For inelastic applications, the model generally requires that the cell analysis (providing the stress-strain relations) and the beam-column analysis be performed simultaneously. Some inelastic applications, which assume that no stress-reversal occurs and hence perform the cell and overall analyses separately, include the studies of box columns [80, 82, 107] and channels [100].

The cell model was also adopted in an approximate elastic nonlinear beam-column analysis [83, 84, 85, 102], featuring an *analytical* solution of a simply supported column undergoing local buckling. The method was approximate because the flexural rigidity was assumed constant along the length, rather than varying as a result of amplitude modulation of the local buckling displacements. In the analysis, Rayleigh-Ritz solutions of the nonlinear von Karman plate equations were obtained to provide the bending moment and the line of action of the axial force for given values of axial force and curvature. The overall analysis consisted of solving the governing differential equation.

<u>Discrete models.</u> The advances in computing technology over the last two decades have made the direct analysis of thin-walled members a feasible proposition. Such analyses involve a discretisation of the member in strips or elements and eigenvalue or nonlinear solutions of, usually, large equation systems.

The semi-analytical finite strip analysis [29] has been applied to the elastic bifurcation analysis of prismatic members [51, 86, 95, 97]. The analysis describes the longitudinal variation of the buckling displacements by harmonic functions and so is limited to simple support conditions. It is, however, a highly efficient computational method. An extension of the method has been described [17], in which various analytic functions are employed to analyse the buckling of plates with boundary conditions other than simple supports. In the context of interaction buckling, the bifurcation analysis is useful for describing the coupling of modes of approximately equal half-wavelength.

The semi-analytical finite strip method has been extended to the geometric nonlinear elastic [47, 53] and inelastic [64] post-local buckling analyses. It has also been extended to the elastic geometric nonlinear analysis of members including local and overall modes [79]. However, no applications to interaction buckling problems were presented in [79].

The spline finite strip method [31] has been employed to analyse the elastic [32, 72, 115] and inelastic [73, 74] bifurcation of prismatic members. The method was particularly suited to analysing boundary conditions other than simple supports and loads acting along the length of the member. The use of splines allowed gradual change of mode shape which lead to a smearing-out [74] of the garland curves typical of the semi-analytical finite strip method. The geometric nonlinear spline finite strip method has been applied to the elastic [69, 70] and inelastic [48] analysis of prismatic members. The spline finite strip method has also been applied to the linear analysis of shells [30, 43].

The finite element method [7, 120] is used increasing to analyse thin-walled structures. Early applications [6, 96] dealt with elastic overall bifurcation using beam elements. The research on overall stability has recently been extended to include large pre-buckling displacements [4, 92, 93].

In [98, 112], a finite element bifurcation analysis is described which includes local and overall deformations. Additional local degrees of freedom are superimposed onto the overall freedoms to include the deflections of component plates. These analyses are restricted to coupling between *local* and overall buckling since the distortional displacements are assumed to vanish at plate junctions. The elastic [16, 19] and inelastic [15] buckling analyses of I- and tee-sections have also been presented, accounting for coupling between overall buckling and web distortion. In these analyses, overall bifurcation and web distortion are assumed to occur at the same half-wavelength. General cross-sectional distortions are considered in the elastic finite element bifurcation analysis of prismatic members described in [41].

A wealth of literature is available on the development of the geometric and material nonlinear analyses of thin-walled structures by full finite element discretisation. Only a few key papers will be mentioned, serving mainly as an introduction to this development. Since shell structures can be modelled as an assembly of flat elements and plated structures can be modelled using shell elements, there is generally no clear distinction between the analyses of plate and shell structures. Rather, the formulation of plate and shell elements rely on the common assumptions that the through-thickness stress vanishes and that plane sections remain plane during the deformation.

Most plane and curved shell elements fall in one of the following three categories: (1) Flat elements that superimpose plane stress and plate-bending elements [8, 26, 34, 40, 91], (2) curved elements formulated by shell theory [3, 62, 103], and (3) degenerated shell elements derived from continuum mechanics by reducing the through-thickness dimension [1, 10, 39, 59, 78, 101, 109, 119]. The formulation of degenerated shell elements is particularly simple being based on three-dimensional continuum mechanics. However, for thin shells, degenerated elements lead to locking problems [10] and consequently, successful implementations require reduced integration and the stabilisation of spurious modes [10, 116].

Shell elements have been implemented in geometric nonlinear analyses, based on total and updated Lagrangian formulations, and material nonlinear analyses. Generally, the implementations are tested against benchmark problems and are, in principle, capable of analysing any kind of elastic and inelastic interaction buckling problem. The applications of finite elements to solve interaction buckling problems are ongoing. The geometric and material nonlinear finite element analyses of a thin-walled box section in bending, a thin-walled hat-section in shear and bending, and thin-walled channel sections in compression and combined bending and torsion have been presented [76, 77].

3. Future needs and developments

3.1 Simulation models

There is no doubt that future developments in the simulation and computational models of interaction buckling will be governed by advances in computing technology. Considering the last two decades, the progress made in developing discrete models far outweighs those made in developing perturbation and cell models, and this trend is likely to continue. The discrete models have been so refined that it is now possible to model closely the behaviour of thin-walled structures, including the interaction of buckling modes and material nonlinearity. As a result, commercial finite element packages, notably Abaqus [58], are used frequently at research institutions to produce design data.

Further developments of *discrete models* are likely to be directed towards: (1) Automated and adaptive mesh generation as well as more user-friendly and powerful pre- and post-processors, (2) refinement of degenerated shell finite elements for enhancing accuracy and efficiency, and (3) new elements. In hot-rolled steel structures standards [5], it is now allowed to use advanced (geometric and material nonlinear) analyses in design. This option is likely also to be made available in standards for thin-walled (or cold-formed) structures. Such development will require finite element programs to become so user-friendly that they can be used safely by design engineers who may not have a detailed knowledge of interaction buckling.

Despite the rapid growth in computing power and the advances in parallel computing [42], several years will pass before full discretisation of large structures in shell elements becomes a realistic proposition. As far as structures composed of thin-walled prismatic members is concerned, there is a need for developing *high order elements* that include cross-sectional distortions. Some investigations have been reported [9, 110] but not thoroughly tested. For members in compression, a beam element with embedded local deformations has been developed [2]. Scope exists for extending this type of element to combinations of compression and bending, and implementing it in frame analyses. Alternatively, beam elements may be based on a softening model [27] in which simplified descriptions (usually bi-linear) of the response of component plates to compression and bending are incorporated in an overall analysis. In the model, the simplified descriptions of component plates are derived from geometric and material nonlinear analyses of individual plates [33, 55, 81, 89].

It may also be possible to base high order beam elements on *generalised beam theory* [104]. An important aspect of the generalised beam theory is that it provides warping functions for open prismatic members undergoing overall and cross-sectional deformations. At present, the theory has been applied to the linear-elastic [36, 104] and elastic bifurcation [37, 75, 105, 106] analyses. Geometric and material nonlinear analyses using generalised beam theory have not yet been developed.

Perhaps the strongest feature of the *perturbation* and *cell models* has been their modal nature and hence ability to separate the effects of participating modes. This feature has been essential in providing an understanding of the phenomena characterising interaction buckling. In view of the fact that it is difficult to separate buckling modes and their effect on the structural behaviour in using full discretisation, the perturbation and cell models are likely to find continuing use in the fundamental research of interaction buckling.

3.2 Cold-formed members

It is widely recognised that cold-formed members are being used increasingly in structural design and probably constitute the fastest growing area in the applications of thin-walled structures. Because of the versatility of the cold-forming process, new and more efficient cross-sections are being developed, notably for use in purlins, storage rack systems and steel framed housing. As these structures are prone to buckling in several modes, they may be imperfection sensitive and buckling patterns are likely to localise. Research into the behaviour of these sections is still at an early stage, and simple models that describe their basic behaviour is needed.

A complexity of purlin and steel framing systems is that the members are attached to sheeting and hence points in the cross-sections are restrained. Several (although similar) elastic theories [44, 56, 57] have been developed for the linear and buckling analyses of such restrained systems. There is a need to further develop these models to include geometric and material nonlinearities.

3.3 Frames

Despite extensive research into the interaction buckling of members, models for analysing frames are lacking. Such models will greatly broaden the scope of interaction buckling studies, as the participating modes need not be in the same member.

Perhaps the greatest difficulty encountered in analysing thin-walled frames is the modelling of joints. Systematic studies are needed for classifying the connections of thin-walled members and proposing models for their load carrying characteristics.

3.4 Reliability analysis

The current move towards generating design data by using numerical models in place of experiments calls for careful attention to random variables. By conducting experiments, random variables are usually represented implicitly, leading to a scatter in the results, or random variables may be considered in the reliability model. For instance, variations in plate thickness and yield stress are considered explicitly in the calibration of most codes of practice, while variations in geometric imperfections are usually assumed to be considered through the scatters of test results.

There is a need to review the use of numerical models for producing design data. Since thin-walled structures are sensitive to geometric imperfections, it is particularly important to ensure that the variation of imperfections is accounted for. This will probably require changes to current reliability models and may have an influence on the use of simulation models.

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