Department of Civil Engineering DTU

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Stiffness of Concrete Slabs

Volume 4

PHD THESIS

BYG • DTU

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1 Preface

This report is prepared as a partial fulfilment of the requirements for obtaining the Ph.D. degree at the Technical University of Denmark.

The work has been carried out at the Department of Structural Engineering and Materials, Technical University of Denmark (BYG•DTU), under the supervision of Professor, dr. techn. M. P. Nielsen.

I would like to thank my supervisor for valuable advise, inspiration and many rewarding discussions and criticism to this work.

Thanks are also due to my co-supervisor M. Sc. Ph.D. Bent Steen Andreasen, RAMBØLL, M. Sc. Ph.D.-student Karsten Findsen, BYG•DTU, M. Sc. Ph.D.-student Lars Z. Hansen, BYG•DTU, M. Sc. Ph.D.-student Thomas Hansen, BYG•DTU and M. Sc. Ph.D.-student João Luís Garcia Domingues Dias Costa, BYG•DTU for their engagement and criticism to the present work and my Ph.D.-project in general.

Finally I would like to thank my wife and family for their encouragement and support.

Lyngby, August 2004

Tim Gudmand-Høyer

2 Summary

This paper treats bending- and torsional stiffness of reinforced concrete slabs subjected to axial forces.
It is assumed that the reinforcement behaves linear elastic in both tension and compression and that the concrete behaves linear elastic in compression and has no tensile strength. Furthermore, it is assumed that the tensile and compressive strength of the reinforcement and the compressive strength of the concrete are not exceeded.
From these assumptions analytical and numerical exact stiffnesses are found for bending and torsion and also for the combination of bending and axial force and the combination of torsion and axial force.
Lower bound solutions have also been investigated and it has been found that such approach may be very useful for the determination of the bending stiffness but it has not lead to satisfactory results for the torsional stiffness.
The relation between bending stiffness and torsional stiffness has been investigated, and it was found that a proposed relation between the stiffnesses given by $D_{xy}=½(D_x D_y)^{½}$ may not be used in general.
Instead it is shown that the torsional stiffness for a given degree of cracking is the same as the bending stiffness for the same degree of cracking for a slab where the reinforcement is placed in the centre. This result is based on a certain way of calculating the degree of cracking, as described in the report, and it is also assumed that the axial forces have the same sign.

3 Resume

Denne rapport behandler emnet bøjnings- og vridningsstivhed for armerede betonplader påvirket med normalkræfter.
Det antages at både armeringen og beton har en lineær elastisk opførelse og at betonen ikke har nogen trækstyrke. Derudover antages det, at armeringsspændingerne ikke overskrider armeringens tryk- og trækstyrke og at betonens trykstyrke ikke overskrides.
Ud fra disse antagelser bestemmes analytiske og numeriske eksakte stivheder for bøjning og vridning samt kombinationerne bøjning og normalkraft og vridning og normalkraft.
Rapporten beskriver hvorledes man kan opstille øvre- og nedreværdiløsninger for stivhederne. Det er vist at nedreværdiløsninger kan være brugbare i tilfældet bøjning mens de ikke er fundet brugbare for vridning med normalkraft.
Sammenhængen mellem bøjnings- og vridningsstivhed er ligeledes beskrevet. Det er i litteraturen forslået at man kan anvende sammenhængen $D_{xy}=½(D_xD_y)^{½}$ men i rapporten er det vist, at dette udtryk ikke kan anvendes generelt.
Til gengæld vises det, at pladens vridningsstivhed for en given revnegrad er den samme som bøjningsstivheden for den samme revnegrad, hvis armeringen er placeret i midten. Dette forudsætter naturligvis en bestemt definition af revnegraden som beskrevet i rapporten, samt at normalkræfterne har samme fortegn.

4 Table of contents

5 Notation

The most commonly used symbols are listed below. Exceptions from the list may appear, but this will then be noted in the text in connection with the actual symbol.

Geometry

h	Depth of a cross-section
h_c	Distance from the bottom face to the centre of the bottom reinforcement
h_c'	Distance from the top face to the centre of the top reinforcement
h_{cx} ,h_{cy}	Distance from the bottom face to the centre of the bottom reinforcement in the *x*- and *y*- direction, respectively
h_{cx}' ,h_{cy}'	Distance from the top face to the centre of the top reinforcement in the *x*- and *y*- direction, respectively
d	Effective depth of the cross-section, meaning the distance from the top face of the slab to the centre of the reinforcement.
A	Area of a cross-section
A_c	Area of a concrete cross-section
A_s	Area of reinforcement per unit length close to the bottom face
A_s'	Area of reinforcement per unit length close to the top face
A_{sx} ,A_{sy}	Area of reinforcement per unit length close to the bottom face in the *x*- and *y*- direction, respectively
A_{sx}', A_{sy}'	Area of reinforcement per unit length close to the top face in the *x*- and *y*- direction, respectively
y_0	Compression zone depth
x, y, z	Cartesian coordinates

Physics

ε	Strain
ε_1, ε_2, ε_x, ε_y	Strain in the 1st principal direction, 2nd principal direction, *x*- direction and *y*- direction, respectively.
γ_{xy}	Shear strain
φ $(=2\gamma_{xy})$	Change of angle

κ	Curvature
κ_x, κ_y, κ_{xy}	Curvatures and torsion in slab.
σ	Normal stress
σ_c	Normal stress in concrete
σ_{cx}, σ_{cy}	Normal stresses in concrete in the *x*- and *y*- direction, respectively.
τ	Shear stress in concrete
ϕ	Reinforcement ratio (for slabs based on total area)
ϕ_x, ϕ_x'	Reinforcement ratio in the *x*-direction for the lower and upper reinforcement, respectively.
ϕ_y, ϕ_y'	Reinforcement ratio in the *y*-direction for the lower and upper reinforcement, respectively.
ρ_{disk}	Reinforcement ratio.
E	Modulus of elasticity
E_s	Modulus of elasticity for the reinforcement
E_c	Modulus of elasticity for the concrete
n	Ratio between the modulus of elasticity for the reinforcement and the modulus of elasticity for the concrete
E_x, E_y	Modulus of elasticity for the slab in the *x*- and *y*-direction, respectively
G_{xy}	Shear-modulus of elasticity for the slab
D_b	Bending stiffness for the slab
D_x,D_y	Bending stiffness for the slab in the *x*- and *y*-direction, respectively
D_{xy}	Torsional stiffness for the slab
$D_{xy,tor\ uncracked}$	Torsional stiffness for the uncracked slab
$D_{xy,tor\ NJN}$	Torsional stiffness for the slab according to N. J. Nielsen
n_x, n_y	Axial force per unit length in the *x*-and *y*-direction, respectively
n_{xy}	Shear force per unit length
m_x, m_y	Bending moment per unit length in the *x*- and *y*-direction, respectively
m_{xy}	Torsional moment per unit length
k_1 to k_6	Constants used in the formulas for the strain field.

6 Introduction

The main purpose of this investigation is to determine the torsional stiffness of reinforced concrete slabs subjected to axial force.
In the investigation of slabs subjected to axial forces the deflection of the slab plays a main role and stiffnesses of the slab are therefore equally important. Previous investigations have shown that in some special cases it may be sufficient to model a slab by a strip model and thereby completely ignore the torsional stiffness. Other investigations use an approximation for the torsional stiffness, based on the bending stiffnesses.
However, in order to calculate slabs subjected to axial force in general the determination of the torsional stiffness is necessary. Therefore, this investigation treats torsional stiffness of slabs subjected to axial force of the same sign in two directions.

The investigation is limited to an investigation on the stiffness dependence of the corresponding actions and the axial force, for instance the bending stiffness dependence on the bending moment and axial forces.
An investigation of bending stiffness dependence on the torsional moment or shear force has not been carried out.
Furthermore, the investigation does not include effects of nonlinear behaviour of the materials and the influence of the tensile strength of concrete.

7 Theory

7.1 Stiffness of slabs

7.1.1 General assumptions

In this section we consider a slab part as the one shown in Figure 7.1. This figure also gives the sign conventions adopted.

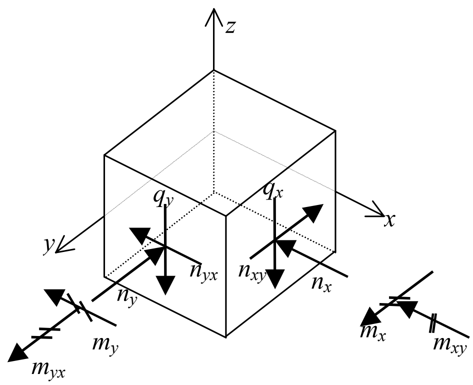


Figure 7.1. Slab part.

Normal forces per unit length n_x and n_y are positive as compression. Bending moments per unit length m_x and m_y are positive when giving tension in the bottom face (the face with the *z*-axis as an inward normal).

Concrete normal stresses σ are normally positive as compression and the corresponding strains positive as shortening.

Reinforcement stresses are normally positive as tension and the corresponding strains positive as elongation.

In this paper we neglect the influence of the shear forces q_x and q_x perpendicular to the slab surfaces.

Regarding constitutive equations it is assumed that the stresses vary linearly with the strains and the concrete has zero tensile strength.

Full compatibility between the longitudinal strains in concrete and reinforcement is assumed i.e. dowel action is neglected. Cracks are smeared out.
For concrete in compression we have therefore:

$$\sigma_c = E_c \varepsilon \tag{7.1.1}$$

For concrete in tension we have:

$$\sigma_c = 0 \tag{7.1.2}$$

For the reinforcement we have:

$$\sigma_s = E_s \varepsilon = n E_c \varepsilon \tag{7.1.3}$$

The compatibility conditions are in general (see [9]):

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
&\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = 2\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\
&\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = 2\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \\
&\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\
&\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} \right) \\
&\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zy}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} \right)
\end{aligned} \tag{7.1.4}$$

If plane stress is assumed (σ_z=τ_{xz}=τ_{yz} =0) the compatibility conditions are reduced to:

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
&\frac{\partial^2 \varepsilon_z}{\partial x^2} = \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = 0
\end{aligned} \tag{7.1.5}$$

In general the compatibility conditions are fulfilled if ε_x, ε_y and γ_{xy} vary linearly with z. This means that ε_x, ε_y and γ_{xy} may be calculated as:

$$\begin{aligned}
\varepsilon_x &= k_1 z + k_2 \\
\varepsilon_y &= k_3 z + k_4 \\
\gamma_{xy} &= k_5 z + k_6
\end{aligned} \tag{7.1.6}$$

where k_1 to k_6 are constants. According to the transformation formulas the relations between the principal strains and the strains in a system rotated the angle α are:

$$\varepsilon_x = \varepsilon_1 \cos^2(\alpha) + \varepsilon_2 \sin^2(\alpha)$$
$$\varepsilon_y = \varepsilon_1 \sin^2(\alpha) + \varepsilon_2 \cos^2(\alpha) \tag{7.1.7}$$
$$\gamma_{xy} = -\frac{1}{2}(\varepsilon_1 - \varepsilon_2)\sin(2\alpha)$$

From this it may be seen that the angle α may be calculated as:

$$\alpha = \frac{1}{2} Arc \tan\left(\frac{2\gamma_{xy}}{\varepsilon_x - \varepsilon_y}\right) \tag{7.1.8}$$

Inserting the functions given in (7.1.6) into (7.1.7) and solving the equations with respect to the principal strains lead to:

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{2}\Big(\big(k_2 - k_4 + (k_1 - k_3)z\big)C + k_2 + k_4 + (k_1 + k_3)z\Big) \\
\varepsilon_2 &= \frac{1}{2}\Big(-\big(k_2 - k_4 + (k_1 - k_3)z\big)C + k_2 + k_4 + (k_1 + k_3)z\Big) \\
&\textit{and} \\
C &= \sqrt{\frac{(k_2 - k_4)^2 + 4k_6^{\,2} + 2\big((k_1 - k_3)(k_2 - k_4) + 4k_5k_6\big)z + \big((k_1 - k_3)^2 + 4k_5^{\,2}\big)z^2}{\big(k_2 - k_4 + (k_1 - k_3)z\big)^2}}
\end{aligned} \tag{7.1.9}$$

This solution of course only has meaning for C real:
This means that in the case of plane stress the problem is reduced to the determination of the factors k_1 to k_6. These factors must be determined so that the equilibrium equations are fulfilled when the constitutive equations are used. However, the calculations do in some cases become quite cumbersome and in some cases the use of numerical calculations is advantageous.

7.2 Beam example

An exact solution requires that both the equilibrium equations, the compatibility equations and the constitutive equations are fulfilled. For slabs the exact solution and thereby the correct stiffness is not always easily found.
However, lower and upper bound solutions may be established and these solutions are much less comprehensive and therefore worth of studying. In this section a beam is calculated as an illustrative example.

7.2.1 Exact stiffness for beams in bending

We consider a reinforced beam subjected to a bending moment only as illustrated in Figure 7.2.

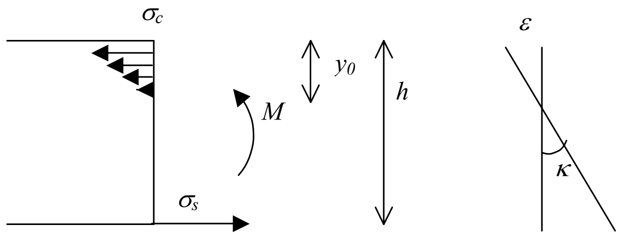


Figure 7.2 Stresses and strains in a beam.

The beam has a rectangular section $b \cdot h$. For simplicity the reinforcement is placed in the bottom face. The projection equation becomes:

$$N = 0 = b\frac{1}{2}\sigma_c y_0 - A_s\sigma_s \tag{7.2.1}$$

The moment equation becomes:

$$M = b\frac{1}{2}\sigma_c y_0\left(\frac{h}{2} - \frac{1}{3}y_0\right) + A_s\sigma_s\frac{h}{2} \tag{7.2.2}$$

where A_s is the cross-section area of the reinforcement.

Assuming that the strains vary linearly as illustrated in Figure 7.2 it is seen that the compatibility equations are fulfilled.
The constitutive relation are also illustrated in Figure 7.2 where the constitutive equations (7.1.1) to (7.1.3) have been used.
Solving the problem leads to the following bending stiffness, D_b, for the beam:

$$\frac{D_b}{E_c bh^3} = \frac{M}{\kappa}\frac{1}{E_c bh^3} = \left(\begin{array}{l} \frac{2}{3}(n\phi)^3 + 2(n\phi)^2 + n\phi \\ -\frac{2}{3}(n\phi)^2\sqrt{(n\phi)^2 + 2n\phi} - \frac{4}{3}n\phi\sqrt{(n\phi)^2 + 2n\phi} \end{array}\right) \tag{7.2.3}$$

Formula (7.2.3) is generally valid for beams with one layer of reinforcement if h is substituted with d, where d is the effective depth.

7.2.2 Lower bound stiffness for beams in bending

Fulfilling only the equilibrium conditions and the constitutive equations and thereby disregarding the compatibility equations establish a lower bound solution. For an elastic system the virtual work equation may be used in order to determine a lower bound stiffness.

Since we are dealing with a “changeable” system in which parts of the section may or may not be effective depending on the cracks the application of standard theorems for elastic systems may be doubtful. This may be compared with a cable structure with cables only effective in tension. However, in what follows we apply the theory of elastic systems without any further comments.

In this example we choose the stress distribution shown in Figure 7.3. This stress distribution has zero stress in the middle, which corresponds to the correct solution for a beam without reinforcement and sufficient tensile strength.

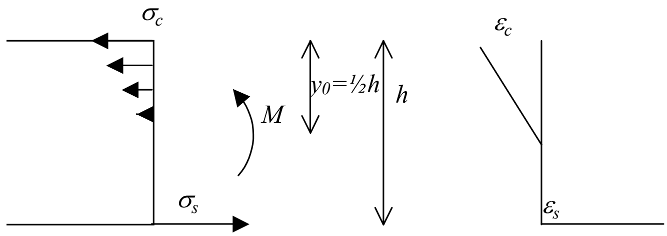


Figure 7.3 Stresses and strains in a beam.

From the projection equation we have:

$$N = 0 = \frac{1}{2} b \frac{1}{2} h\sigma_c - \phi bh\sigma_s \tag{7.2.4}$$

The moment equation becomes:

$$M = \frac{1}{2} b \frac{1}{2} h\sigma_c \frac{2}{3}\frac{h}{2} + \phi bh\sigma_s \frac{h}{2} \tag{7.2.5}$$

Combining these equilibrium equations with the constitutive equations leads to the determination of the strain field. The strains are illustrated in Figure 7.3 and the main values become:

$$\varepsilon_c = \frac{24}{5}\frac{M}{h^2 bE_c} \tag{7.2.6}$$

$$\varepsilon_s = \frac{6}{5}\frac{M}{h^2 bE_c \phi n} \tag{7.2.7}$$

With the strains given, it is possible to find the stiffness using the virtual work equation, which for a beam in bending may be written:

$$M^1\kappa = \int_A \sigma^1 \varepsilon dA \tag{7.2.8}$$

Here M^1 is a fictitious (virtual) moment and σ^1 is a stress distribution equivalent to the fictitious moment. The stresses σ^1 may therefore be chosen freely as long as the moment and projection equations are fulfilled. A simple solution is to apply a moment of magnitude 1 (M^1=1) and choose a stress distribution similar to the one valid for the real moment M. In this case we get the stress and strain distributions shown in Figure 7.4.

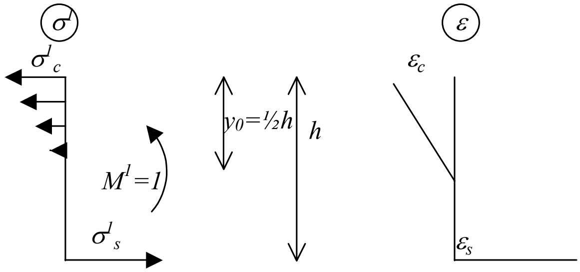


Figure 7.4 Stresses and strains in a beam.

The stresses may be found using the projection equation and the moment equation, which leads to:

$$\sigma^1{}_c = \frac{24}{5}\frac{1}{h^2 b} \tag{7.2.9}$$

$$\sigma^1{}_s = \frac{6}{5}\frac{1}{h^2 b\phi} \tag{7.2.10}$$

Using the virtual work equation leads to the following stiffness:

$$\begin{aligned} &1\kappa = b\int_0^h \sigma^1 \varepsilon dy \Leftrightarrow \\ &\kappa = b\frac{1}{3}\frac{h}{2}\sigma_c{}^1\varepsilon_c + \phi bh\sigma_s{}^1\varepsilon_s \Leftrightarrow \\ &\frac{M}{\kappa}\frac{1}{E_c bh^3} = \frac{25\phi n}{96\phi n + 36} \end{aligned} \tag{7.2.11}$$

This result is compared with an upper bound and the exact solution below.

7.2.3 Upper bound stiffness for beams in bending

Fulfilling only the compatibility conditions and the constitutive equation and thereby disregarding the equilibrium equations, establish an upper bound solution. Using the virtual work equation it is possible to determine an upper bound stiffness.
In this example we choose the strain distribution shown in Figure 7.3. This strain distribution has zero strains in the middle, which corresponds to the correct solution for a beam without reinforcement and with sufficient tensile strength.

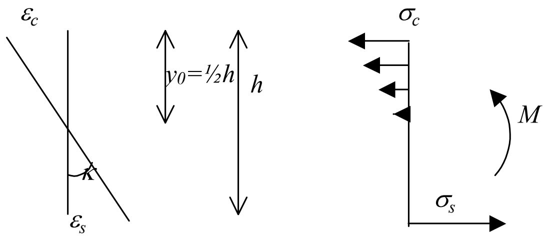


Figure 7.5 Stresses and strains in a beam.

The constitutive equations lead to the following stresses:

$$\sigma_c = \varepsilon_c E_c = \kappa \frac{h}{2} E_c \tag{7.2.12}$$

$$\sigma_s = \varepsilon_s E_s = \kappa \frac{h}{2} n E_c \tag{7.2.13}$$

With the stresses given it is possible to find the stiffness using the virtual work equation. In this case we apply a virtual relative rotation and thus we get:

$$M\kappa^1 = \int_A \sigma\varepsilon^1 dA \tag{7.2.14}$$

Here κ^1 is the virtual rotation and ε^1 is a strain distribution corresponding to the virtual rotation. This distribution may be chosen freely as long as the compatibility conditions are fulfilled. A simple solution is to apply a rotation of magnitude 1 (κ^1=1) and choose a strain distribution similar to the one valid for the real rotation. In this case we get the stress and strain distribution as shown in Figure 7.6.

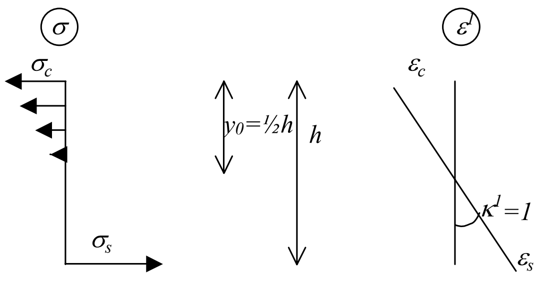


Figure 7.6 Stresses and strains in a beam.

The strains are:

$$\varepsilon_c^{\,1} = \kappa^1 \frac{h}{2} = \frac{h}{2} \tag{7.2.15}$$

$$\varepsilon_s^{\,1} = \kappa^1 \frac{h}{2} = \frac{h}{2} \tag{7.2.16}$$

Using the virtual work equation leads to the following stiffness:

$$\begin{aligned} M1 &= b\int_0^h \sigma\varepsilon^1 dy \Leftrightarrow \\ M &= b\frac{1}{3}\frac{h}{2}\sigma_c\varepsilon_c{}^1 + \phi bh\sigma_s\varepsilon_s{}^1 \Leftrightarrow \\ \frac{M}{\kappa}\frac{1}{E_c bh^3} &= \frac{1}{24} + \frac{1}{4}\phi\text{n} \end{aligned} \tag{7.2.17}$$

7.2.4 Comparison of stiffnesses

Having the exact stiffness, a lower bound stiffness and an upper bound stiffness given by equations (7.2.3), (7.2.11) and (7.2.17), respectively, it is possible to compare the results and evaluate the accuracy. This is done in Figure 7.7 and Figure 7.8. The deviation is defined as the absolute value of the difference between the solutions over the exact stiffness.

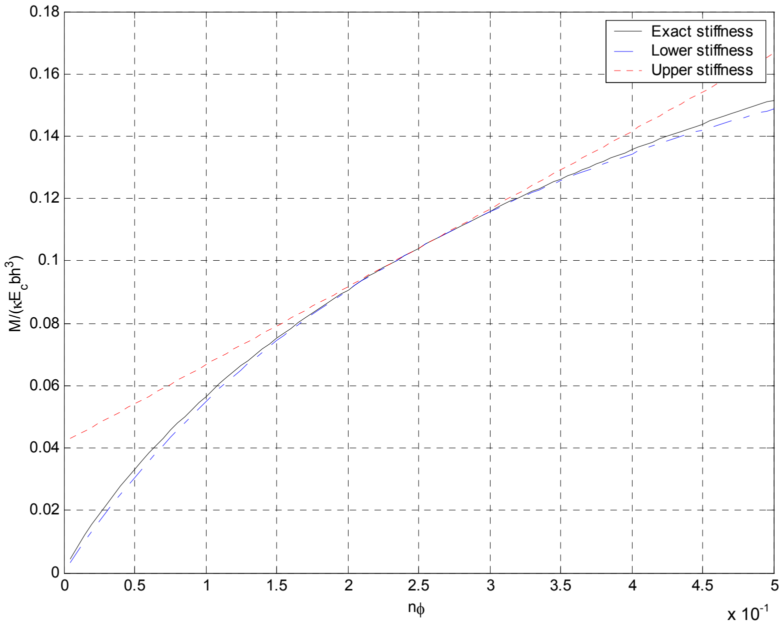


Figure 7.7 Exact, lower and upper stiffness for a beam.

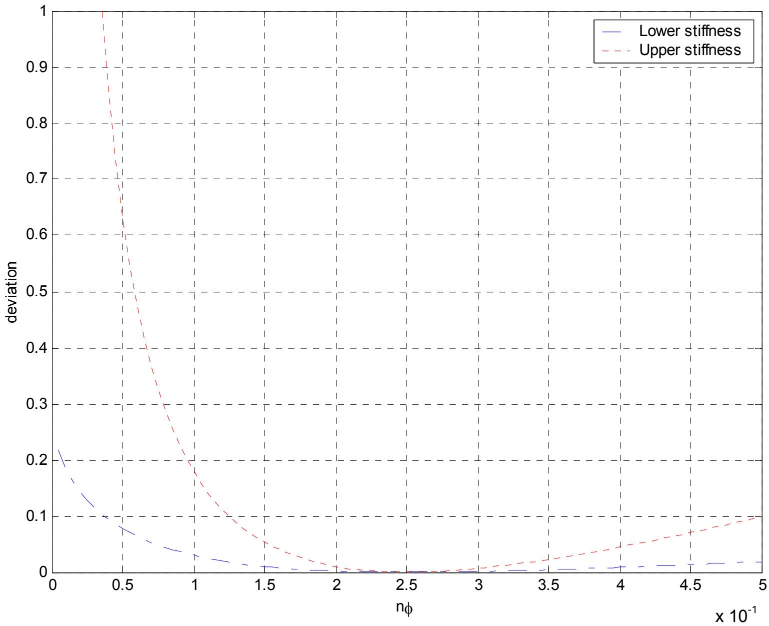


Figure 7.8 Deviations for upper and lower stiffness.

It appears that even a very simple lower bound solution leads to a reasonable result for most reinforcement ratios. The upper bound solution on the other hand overestimates the stiffness for small reinforcement ratios. Furthermore, it should be noted that the lower bound solution gives a dependency on the reinforcement ratio ($n\phi$) that is very similar to the correct solution.

Whether solutions of this kind are sufficiently accurate or not does of course depend on the context in which they are used. Nevertheless, these simple calculations indicate that such methods may be used to calculate stiffnesses in more complicated cases such as slabs with axial forces and torsion.

The choice of stress or strain fields affects the resulting stiffness. To get an idea of this influence we calculate a lower bound stiffness for the beam choosing a statically admissible stress field as shown in Figure 7.9 and a virtual stress field as shown in Figure 7.10.

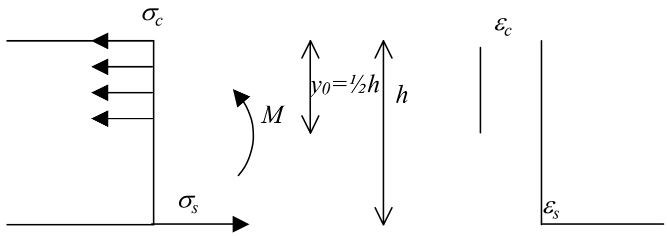


Figure 7.9 Stresses and strains in a beam.

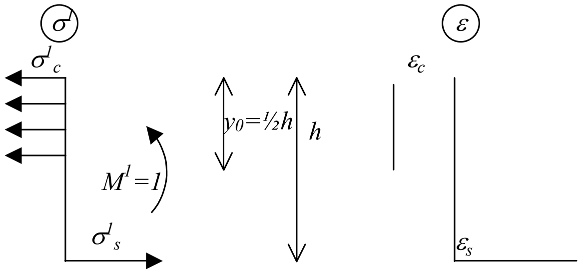


Figure 7.10 Stresses and strains in a beam.

By similar calculations as those described in section 7.2.2 we find the stiffness:

$$\frac{M}{\kappa}\frac{1}{E_c bh^3} = \frac{9\phi n}{32\phi n + 16} \tag{7.2.18}$$

The dependency on the reinforcement ratio ($n\phi$) may be seen in Figure 7.11 where it has been compared with the previous solutions. The present solution is marked Lower 2 in the plots.

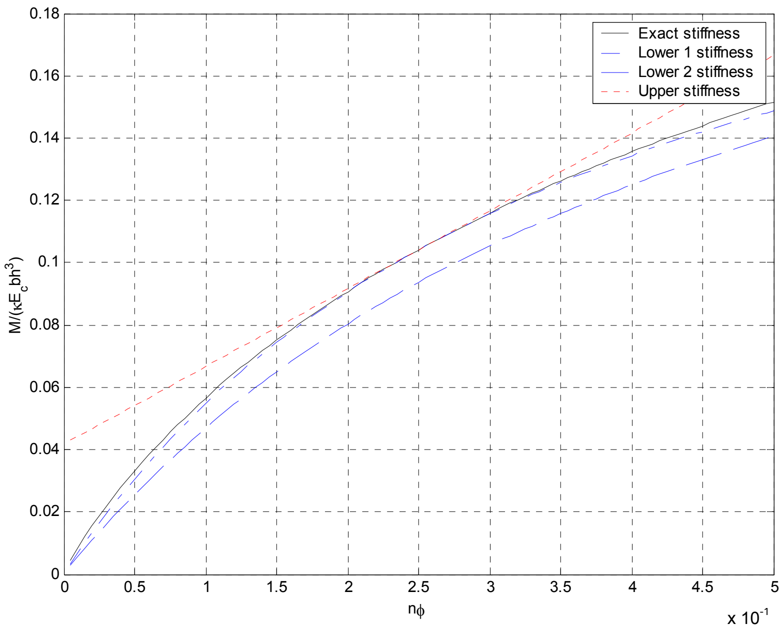


Figure 7.11 Exact, lower1, lower 2 and upper bound stiffness for a beam.

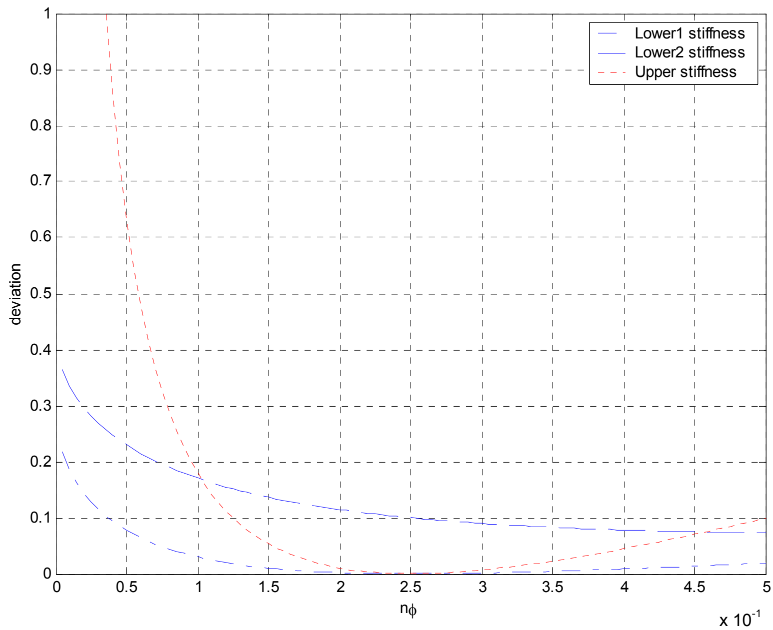


Figure 7.12 Deviations for upper, lower 1 and lower 2 bound stiffnesses.

As seen, even a stress field quite different from the exact stress field leads to reasonable results.

7.3 Slab stiffness

In this chapter the geometry is defined according to Figure 7.13.

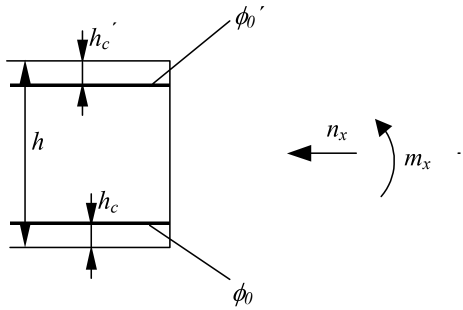


Figure 7.13. Slab with axial force.

7.3.1 Bending stiffness for slabs without torsion

For slabs subjected to bending only the bending stiffness may be calculated as for a beam and the calculations are easily made. However, even for a slab subjected to bending and axial force the calculations start to become cumbersome. The equilibrium equations are:

$$n_x = \frac{1}{2}\sigma_c y_0 - \left(\frac{\frac{\sigma_c}{E_c}}{y_0} nE_c \left(h - y_0 - h_c \right) \right) \phi h - \left(-\frac{\frac{\sigma_c}{E_c}}{y_0} nE_c \left(y_0 - h_c' \right) \right) \phi' h \tag{7.3.1}$$

$$\begin{aligned} m_x = {} & \frac{1}{2}\sigma_c y_0 \left(\frac{h}{2} - \frac{1}{3} y_0 \right) + \left(\frac{\frac{\sigma_c}{E_c}}{y_0} nE_c \left(h - y_0 - h_c \right) \right) \phi h \left(\frac{h}{2} - h_c \right) \\ & - \left(-\frac{\frac{\sigma_c}{E_c}}{y_0} nE_c \left(y_0 - h_c' \right) \right) \phi' h \left(\frac{h}{2} - h_c' \right) \end{aligned} \tag{7.3.2}$$

These may be written in dimensionless form as:

$$\frac{n_x}{\sigma_c h}=\frac{\frac{1}{2}\left(\frac{y_0}{h}\right)^2+\left(n\phi'+n\phi\right)\frac{y_0}{h}+n\phi\left(\frac{h_c}{h}-1\right)+n\phi'\left(-\frac{h_c'}{h}\right)}{\frac{y_0}{h}} \tag{7.3.3}$$

$$\frac{m_x}{\sigma_c h^2}=\frac{-\frac{1}{6}\left(\frac{y_0}{h}\right)^3+\frac{1}{4}\left(\frac{y_0}{h}\right)^2+\left(n\phi\left(\frac{h_c}{h}-\frac{1}{2}\right)+n\phi'\left(\frac{1}{2}-\frac{h_c'}{h}\right)\right)\frac{y_0}{h}+n\phi\left(\frac{1}{2}-\frac{3}{2}\frac{h_c}{h}+\left(\frac{h_c}{h}\right)^2\right)+n\phi'\left(-\frac{1}{2}\frac{h_c'}{h}+\left(\frac{h_c'}{h}\right)^2\right)}{\frac{y_0}{h}} \tag{7.3.4}$$

The depth of the compression zone y_0 may be found for a given combination of m_x and n_x by solving these equations. It appears that y_0 is constant for a given ratio of n_xh/m_x and may therefore be written as a function of this ratio. The problem leads to a 3rd-degree equation:

$$\begin{aligned}0 = &\left(\frac{y_0}{h}\right)^3+\left(\frac{3}{\frac{n_xh}{m_x}}-\frac{3}{2}\right)\left(\frac{y_0}{h}\right)^2\\ &+\left(\frac{6\left(n\phi'+n\phi\right)}{\frac{n_xh}{m_x}}-6\left(n\phi\left(\frac{h_c}{h}-\frac{1}{2}\right)+n\phi'\left(\frac{1}{2}-\frac{h_c'}{h}\right)\right)\right)\frac{y_0}{h}\\ &+6\left(\frac{n\phi\left(\frac{h_c}{h}-1\right)+n\phi'\left(-\frac{h_c'}{h}\right)}{\frac{n_xh}{m_x}}\right)-6\left(n\phi\left(\frac{1}{2}-\frac{3}{2}\frac{h_c}{h}+\left(\frac{h_c}{h}\right)^2\right)+n\phi'\left(-\frac{1}{2}\frac{h_c'}{h}+\left(\frac{h_c'}{h}\right)^2\right)\right)\end{aligned} \tag{7.3.5}$$

It may be shown that the discriminant is positive if $0<y_0/h<1$ and only one real solution exists to the problem. The solution is extensive and may be seen in appendix 10.1.
In the special case of only one layer of reinforcement placed in centre we get:

$$0=\left(\frac{y_0}{h}\right)^3+3\left(\frac{m_x}{n_xh}-\frac{1}{2}\right)\left(\frac{y_0}{h}\right)^2+6n\phi\frac{m_x}{n_xh}\frac{y_0}{h}-3n\phi\frac{m_x}{n_xh} \tag{7.3.6}$$

σ_c may now be determined from the projection equation (7.3.3) giving:

$$\sigma_c = \frac{n_x \dfrac{y_0}{h}}{\left(\dfrac{1}{2}\left(\dfrac{y_0}{h}\right)^2 + \left(n\phi' + n\phi\right)\dfrac{y_0}{h} + n\phi\left(\dfrac{h_c}{h} - 1\right) + n\phi'\left(-\dfrac{h_c'}{h}\right)\right)h}$$

$$\frac{\sigma_c h}{n_x} = \frac{\dfrac{y_0}{h}}{\left(\dfrac{1}{2}\left(\dfrac{y_0}{h}\right)^2 + \left(n\phi' + n\phi\right)\dfrac{y_0}{h} + n\phi\left(\dfrac{h_c}{h} - 1\right) + n\phi'\left(-\dfrac{h_c'}{h}\right)\right)} \tag{7.3.7}$$

Knowing the depth of the compression zone, the bending stiffness $D=D_x$ may be calculated as:

$$D_x = \frac{m_x}{\kappa_x} = \frac{\dfrac{m_x}{\sigma_c h^2}\sigma_c h^2}{\dfrac{\dfrac{\sigma_c}{E_c}}{y_0}} = \frac{m_x}{\sigma_c h^2}\frac{y_0}{h}h^3 E_c = \frac{m_x}{n_x h}\frac{n_x}{\sigma_c h}\frac{y_0}{h}h^3 E_c$$

$$\frac{D_x}{h^3 E_c} = \frac{1}{\dfrac{n_x h}{m_x}}\frac{n_x}{\sigma_c h}\frac{y_0}{h} \tag{7.3.8}$$

It is seen that besides from the physical properties of the slab the bending stiffness only depends on the $n_x h/m_x$ ratio in combination with the sign of n_x. The limits for $n_x h/m_x$ corresponding to $y_0=h$ and $y_0=0$, respectively, become:

$$\frac{n_x h_{(y_0=h)}}{m_x} = \frac{\dfrac{1}{2} + n\phi\dfrac{h_c}{h} + n\phi'\left(1 - \dfrac{h_c'}{h}\right)}{\dfrac{1}{12} + \left(-\dfrac{1}{2}\dfrac{h_c}{h} + \left(\dfrac{h_c}{h}\right)^2\right)n\phi + \left(\dfrac{1}{2} - \dfrac{3}{2}\dfrac{h_c'}{h} + \left(\dfrac{h_c'}{h}\right)^2\right)n\phi'} \tag{7.3.9}$$

$$\frac{n_x h_{(y_0=0)}}{m_x} = \frac{-2\left(\phi\left(1 - \dfrac{h_c}{h}\right) + \phi'\dfrac{h_c'}{h}\right)}{\phi\left(1 - 3\dfrac{h_c}{h} + 2\left(\dfrac{h_c}{h}\right)^2\right) + \phi'\left(-\dfrac{h_c'}{h} + 2\left(\dfrac{h_c'}{h}\right)^2\right)} \tag{7.3.10}$$

If the $n_x h/m_x$ ratio is larger than the limit for $y_0=h$ the stiffness of the slab equals the uncracked stiffness given by:

$$\frac{D_{x(y_0=h)}}{h^3 E_c} = \frac{1}{12} + \left(\left(\frac{h_c}{h}\right)^2 - \frac{1}{2}\frac{h_c}{h}\right)n\phi + \left(\frac{1}{2} - \frac{3}{2}\frac{h_c'}{h} + \left(\frac{h_c'}{h}\right)^2\right)n\phi' \tag{7.3.11}$$

If the $n_x h/m_x$ ratio is lower than the limit for $y_0=0$ the stiffness of the slab equals the stiffness of the reinforcement only given by:

$$\frac{D_{x_{(y_0=0)}}}{h^3 E_c} = \left(\frac{1}{2} + \left(\frac{h_c}{h}\right)^2 - \frac{3}{2}\frac{h_c}{h}\right) n\phi + \left(\left(\frac{h_c'}{h}\right)^2 - \frac{1}{2}\frac{h_c'}{h}\right) n\phi' \tag{7.3.12}$$

Calculations for some slabs with the same reinforcement in the top and bottom ($\phi = \phi'$) are shown in Figure 7.14. In Figure 7.15 the variation of the depth of the compression zone may be seen.

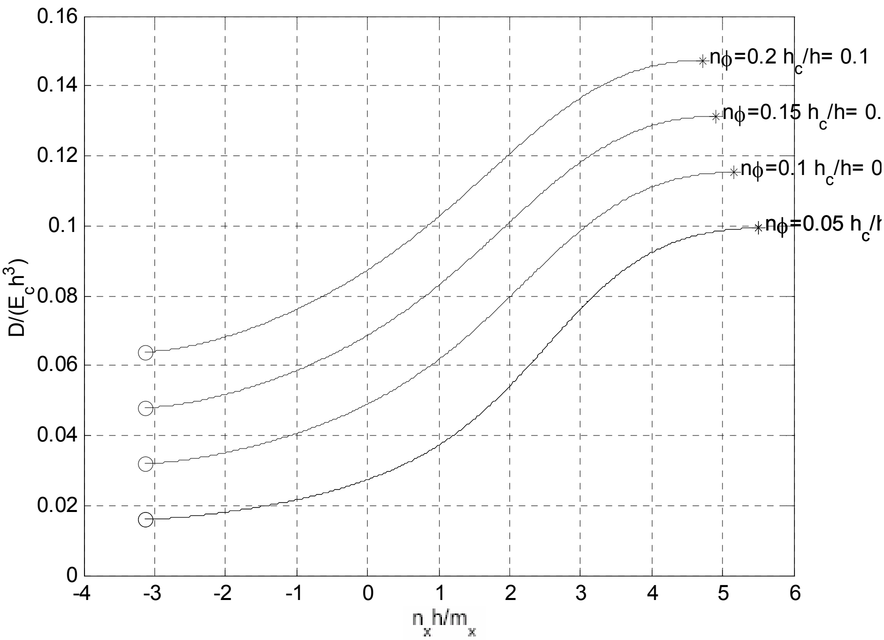


*Figure 7.14 Bending stiffness for slabs subjected to bending and axial force. The * point and the O point mark the uncracked and the fully cracked stiffness given by formulas (7.3.11) and (7.3.12), respectively.*

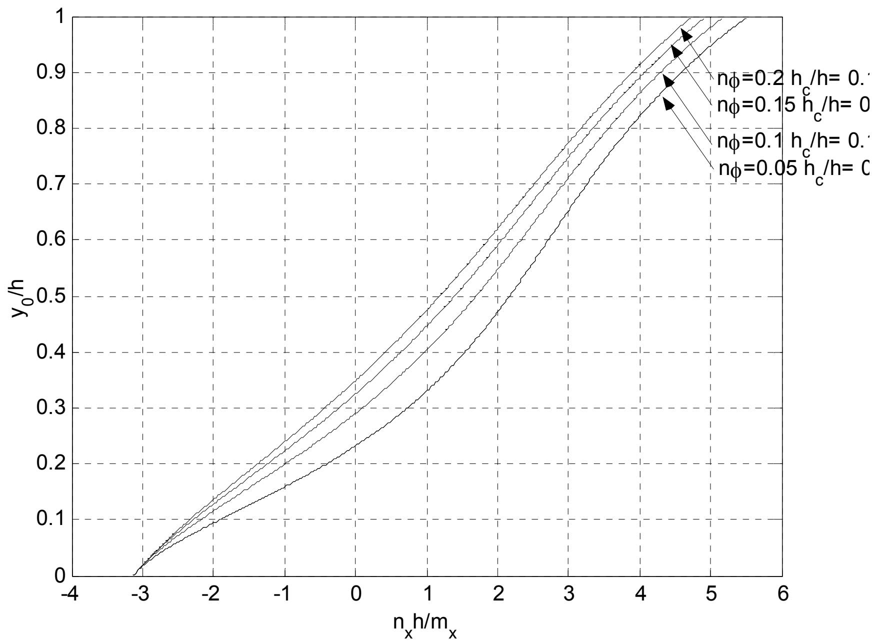


Figure 7.15 Variation of the depth of the compression zone with the axial force.

For slabs reinforced only in the bottom face the variation is quite different. This may be seen in Figure 7.16 and Figure 7.17 showing the stiffness and depth of the compression zone for different slabs.

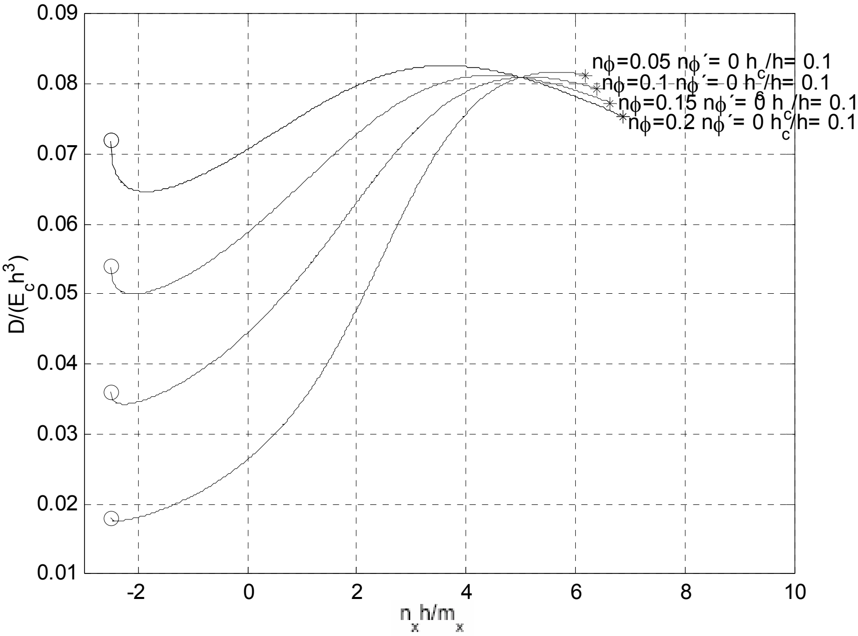


*Figure 7.16 Bending stiffness for slabs subjected to bending and axial force. The * point and the O point mark the uncracked and the fully cracked stiffness given by formulas (7.3.11) and (7.3.12), respectively.*

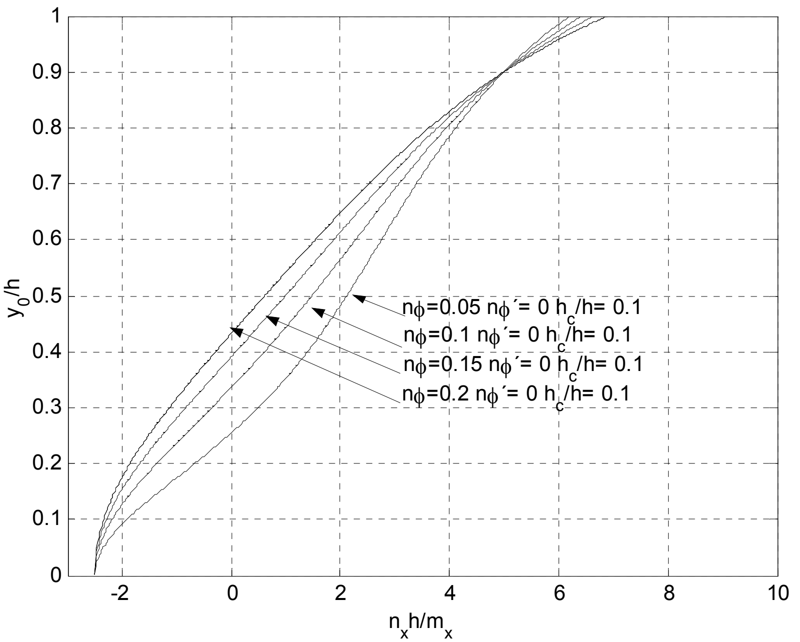


Figure 7.17 Variation of the depth of the compression zone with the axial force.

From Figure 7.16 it appears that the slab is actually a bit stiffer when it is a bit cracked compared to the uncracked situation. This may seem somewhat strange since it is well known that the moment of inertia of an uncracked cross-section is larger than the moment of inertia of a cracked cross-section. Also the fact that a fully cracked cross-section has some stiffness may seem strange. However, the explanation is to be found in the definition of the reference point for the axial force. In this paper the reference point is kept constant at a distance of *h/2* from the top surface. Compared to the usual use of Naviers formula where the reference point of the axial force is at the centre of gravity of the transformed cross-section, the definition used in this paper results in an additional moment.

The reason for choosing this, apparently strange, definition is simply the fact that it makes the calculations somewhat simpler.

The bending stiffness is, in general, a function of the degree of cracking. For a given axial force the moment at the transition state between cracked and uncracked ($y_0=h$) section is named the cracking moment. The ratio between the applied moment and the cracking moment may be used as a degree of cracking, thereby making it possible to express the stiffness as a function of the degree of cracking.

The cracking moment may be found from equation (7.3.9) as:

$$m_{x,crack} = \frac{n_x h_{(y_0=h)}\left(\frac{1}{12}+\left(-\frac{1}{2}\frac{h_c}{h}+\left(\frac{h_c}{h}\right)^2\right)n\phi+\left(\frac{1}{2}-\frac{3}{2}\frac{h_c'}{h}+\left(\frac{h_c'}{h}\right)^2\right)n\phi'\right)}{\frac{1}{2}+n\phi\frac{h_c}{h}+n\phi'\left(1-\frac{h_c'}{h}\right)} \qquad (7.3.13)$$

In the special case with only one layer of reinforcement placed in the centre of the slab we get:

$$m_{x,crack} = \frac{1}{6}\frac{n_x h_{(y_0=h)}}{1+n\phi} \qquad (7.3.14)$$

In Figure 7.18 the stiffness as a function of the degree of cracking is shown for a slab with one layer of reinforcement in the centre. In Figure 7.19 the stiffness is shown for two layers and in Figure 7.20 for one layer placed at a distance of $h_c=0.1h$.

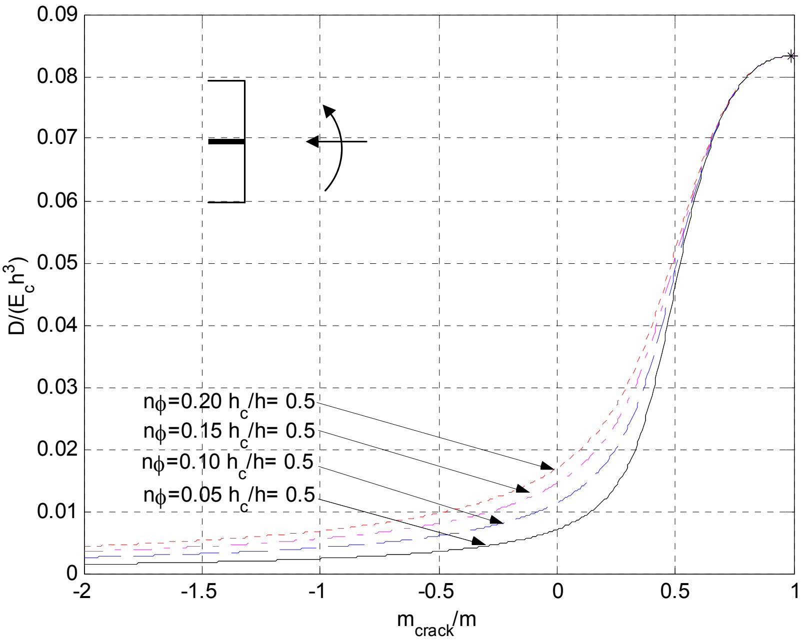


Figure 7.18 Bending stiffness as a function of the ratio cracking moment over moment. h_c=½h and one layer of reinforcement.

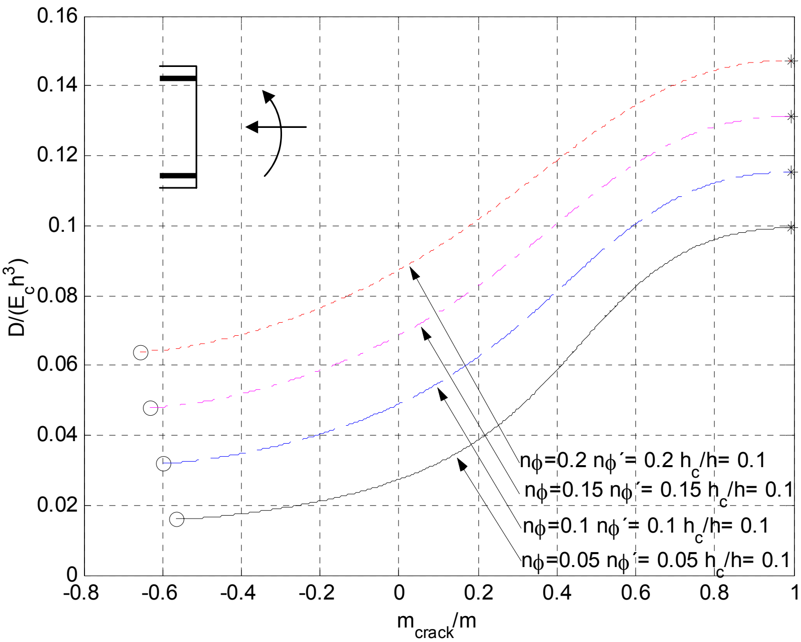


Figure 7.19 Bending stiffness as a function of the ratio cracking moment over moment. h_c= h_c' =0.1h and two layers of reinforcement.

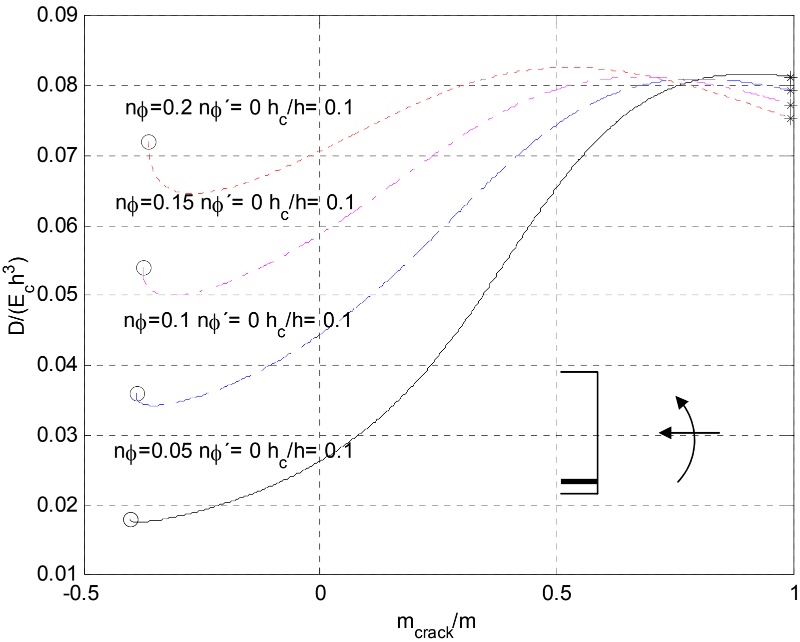


Figure 7.20 Bending stiffness as a function of the ratio cracking moment over moment. h_c= 0.1h and one layer of reinforcement.

The diagrams shown in Figure 7.18 to Figure 7.20 may be used directly to calculate the stiffness.

This way of determining the stiffness as a function of the degree of cracking may seem as unnecessary playing around with already existing knowledge. However, it will later be shown that the approach has some advantages.

7.3.2 Numerical calculation of the stiffness

As mentioned previously in section 7.1.1 the determination of the strains and stresses may be reduced to the problem of determining the six constants k_1 to k_6. This may be done numerically and in this paper a program with the structure described in Appendix 10.2 is used. The main content of this program is the Newton-Raphson method. A guess on the six constants k_1 to k_6 is made and then the Newton-Raphson method is used until a satisfactory result is achieved (see [7]). From the applied forces and moments the stiffnesses may be calculated as:

$$E_x = \frac{n_x}{k_2} \qquad E_y = \frac{n_y}{k_4} \qquad G_{xy} = \frac{n_{xy}}{k_6}$$
$$D_x = \frac{m_x}{k_1} \qquad D_y = \frac{m_y}{k_3} \qquad D_{xy} = \frac{m_{xy}}{k_5} \tag{7.3.15}$$

The program makes it possible to determine the stiffness for any combination of actions.

7.3.3 Torsional stiffness without bending

One of the most interesting investigations is the change of torsional stiffness as a function of the axial force.

In the case of pure torsion of an isotropic slab, N. J. Nielsen has shown that the exact stiffness may be calculated as (see [1] or [11]):

$$D_{xy,torNJN} = E_c h^3 \frac{1}{4}\left(\frac{a}{h}\right)^2 \left(1 - \frac{2}{3}\frac{a}{h}\right)$$
$$where \qquad (7.3.16)$$
$$\frac{a}{h} = 2\frac{A_s E_s}{hE_c}\left(-1 + \sqrt{1 + \frac{1}{2\frac{A_s E_s}{hE_c}}}\right)$$

This may also be written as:

$$\frac{D_{xy}}{E_c h^3} = \begin{pmatrix} \frac{2}{3}(n\phi)^3 + (n\phi)^2 + \frac{1}{4}n\phi \\ -\frac{2}{3}(n\phi)^2\sqrt{(n\phi)^2 + 2n\phi} - \frac{2}{3}n\phi\sqrt{(n\phi)^2 + 2n\phi} \end{pmatrix} \qquad (7.3.17)$$

It appears that the torsional stiffness is independent of the position of the reinforcement, still being symmetrical with respect to the slab middle surface.

In the uncracked case we have when Poisson's ratio is zero (see [11]):

$$D_{xy,tor\_uncracked} = \frac{1}{12}E_c h^3 \qquad (7.3.18)$$

If an axial force in both the *x*- and *y*- direction is applied gradually, the torsional stiffness changes from the cracked stiffness to the uncracked stiffness. This means that the stiffness depends on how cracked the cross-section is. In order to quantify the degree of cracking it is practical to introduce a ratio of some kind.

In this case the transition point between the cracked and the uncracked state is used as a reference state. This state is indexed cracked. The state is found from the formulas for an uncracked cross-section corresponding to the state where one of the principal strains is zero. This approach is similar to the determination of the bending stiffness.

In the uncracked case with axial force and torsion we have the following longitudinal strains in the middle surface of the slab (positive as shortening):

$$\varepsilon_x = \frac{n_x}{E_c h\left(1+2\phi_x n\right)}$$
$$\varepsilon_y = \frac{n_y}{E_c h\left(1+2\phi_y n\right)} \tag{7.3.19}$$

And we have the torsion:

$$\kappa_{xy} = \frac{12 m_{xy}}{E_c h^2} \tag{7.3.20}$$

Using the transformation formulas it is possible to determine the principal strains and the angle between the first principal axis and the *x*-axis. However, the formulas are quite extensive and will not be given here. The principal strains are functions of the position (z). Setting the position equal to a point in the top face of the slab and setting the principal strain equal to zero lead to an equation from which the torsional cracking moment may be found. In the case of torsion and axial force we get:

$$m_{xy,crack} = \frac{h\sqrt{n_y\left(1+2\phi_y n\right)\left(1+2\phi_x n\right)n_x}}{6\left(1+2\phi_y n+2\phi_x n+4\phi_x n\phi_y n\right)} \tag{7.3.21}$$

Torsional moments larger than this lead to a more cracked cross-section and it is therefore reasonable to use the ratio between the applied moment and the cracking moment as a measure of the degree of cracking.
Such ratio for the degree of cracking is only meaningful if the cross-section is uncracked when no torsion is applied. This means that if one of the axial forces is negative (tension) this measure does not have any physical meaning. Similarly if both axial forces are negative no physical meaning may be attributed to the cracking moment but we do have a solution and thereby a reference point. If we redefine the torsional cracking moment so that it changes sign for negative axial force (tension) in both directions we get:

$$m_{xy,crack} = sign\left(n_y\right)\frac{h\sqrt{n_y\left(1+2\phi_y n\right)\left(1+2\phi_x n\right)n_x}}{6\left(1+2\phi_y n+2\phi_x n+4\phi_x n\phi_y n\right)} \tag{7.3.22}$$

Using this definition we may now calculate numerically the torsional stiffness as a function of the degree of cracking, i.e. the torsional cracking moment over the applied torsional moment.
Figure 7.21 shows the results of the calculations for a slab with a degree of reinforcement of $n\phi=0.05$.

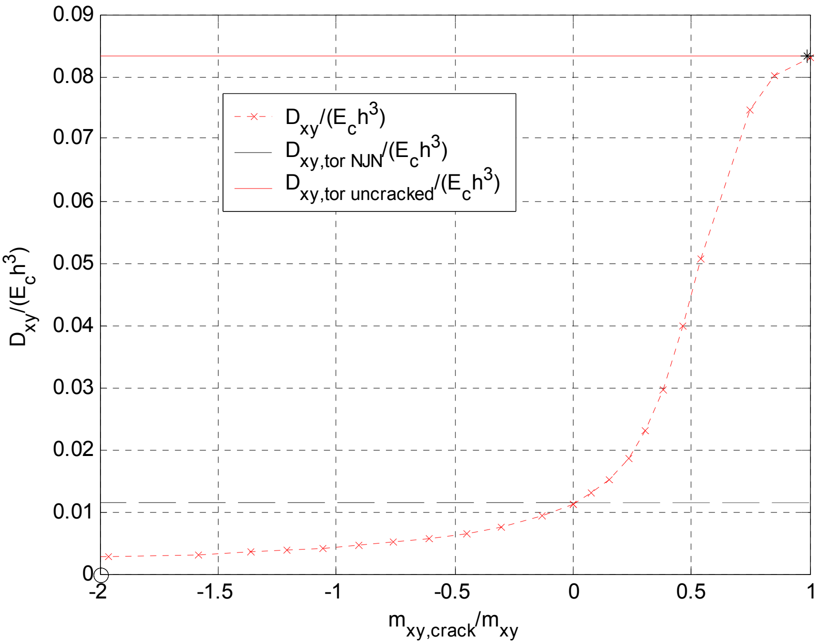


Figure 7.21 Torsional stiffness for a slab subjected to axial force in two directions, $n_y=½n_x$. The degree of reinforcement is $n\phi=0.05$ and the slab is isotropic.

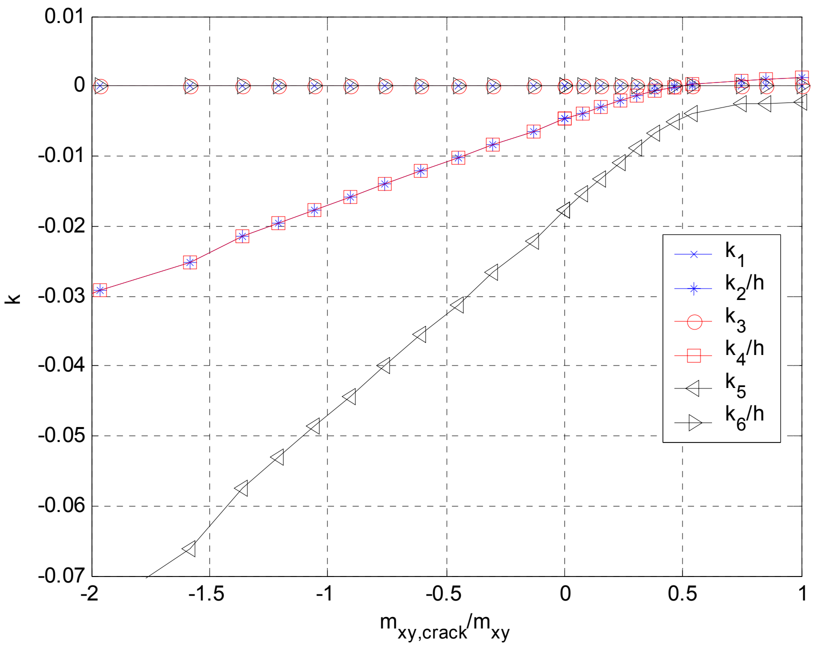


Figure 7.22 Variation of k_1 to k_6 for a slab subjected to axial force in two directions. The degree of reinforcement is $n\phi$=0.05 and the slab is isotropic.

In Figure 7.22 the variation of k_1 to k_6 is shown as a function of the axial force. From this plot it appears that k_1, k_3 and k_6 are zero. This is quite logical since the slab is isotropic and therefore, no curvature in the x- and y- directions is to be expected. k_6 determines the point of zero shear strain and since there are no shear forces it is logical that the relative rotation takes place around the centre of the slab. Furthermore, we have k_2 equal to k_4 which is also logical since the reinforcement is the same in the x- and y- directions.

Considerations of this kind may be used directly to find a general stiffness formula for the isotropic case. We assume that k_1, k_3 and k_6 are zero, and that k_2 equals k_4.

Introducing these values into (7.1.6) we get:

$$\begin{aligned} \varepsilon_x &= k_2 \\ \varepsilon_y &= k_2 \\ \gamma_{xy} &= k_5 z \end{aligned} \tag{7.3.23}$$

From (7.1.8) it follows that the angle α is:

$$\alpha = \frac{1}{2} Arc\tan\left(\frac{2\gamma_{xy}}{\varepsilon_x - \varepsilon_y}\right) = \frac{\pi}{4} \tag{7.3.24}$$

The principal strains may then be calculated from (7.1.7) as:

$$\left.\begin{aligned} k_2 &= \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 \\ k_2 &= \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 \\ k_5 z &= -\frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right) \end{aligned}\right\} \Leftrightarrow \begin{cases} \varepsilon_1 = k_2 - k_5 z \\ \varepsilon_2 = k_2 + k_5 z \end{cases} \tag{7.3.25}$$

This results in the stresses and strains shown in Figure 7.23.

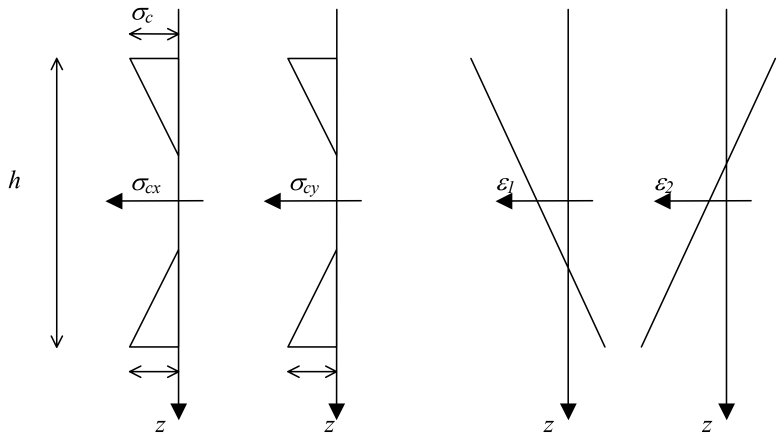


Figure 7.23 Stresses and strains for pure torsion.

The main stresses illustrated in Figure 7.23 become:

$$\begin{aligned} \sigma_c &= \frac{1}{2}E_c\left(k_2 + \frac{1}{2}k_5 h\right) \\ \sigma_s &= nE_c k_2 \end{aligned} \tag{7.3.26}$$

Using this strain field in the equilibrium equations and simplifying the expressions we get:

$$m_x = 0 \tag{7.3.27}$$

$$m_y = 0 \tag{7.3.28}$$

$$m_{xy} = -\frac{1}{24}\left(1 + 2\frac{k_2}{k_5 h}\right)^2 k_5\left(-1 + \frac{k_2}{k_5 h}\right)E_c h^2 \tag{7.3.29}$$

$$m_{xy} = \frac{1}{24}\left(h - 2\frac{k_2}{k_5}\right)^2 E_c k_5\left(h + \frac{k_2}{k_5}\right) \tag{7.3.30}$$

$$n_{xy} = 0 \tag{7.3.31}$$

$$n_x = n_y = \frac{m_{xy}}{h} \frac{3\left(4\frac{k_2}{k_5 h}+1+4\left(\frac{k_2}{k_5 h}\right)^2+16\phi n\frac{k_2}{k_5 h}\right)}{\left(1+2\frac{k_2}{k_5 h}\right)^2\left(1-\frac{k_2}{k_5 h}\right)} \tag{7.3.32}$$

The stiffness becomes:

$$\frac{D_{xy}}{E_c h^3} = \frac{1}{24}\left(1+2\frac{k_2}{k_5 h}\right)^2\left(1-\frac{k_2}{k_5 h}\right) \tag{7.3.33}$$

It is seen that the strain field assumed leads to the stress field required and the stiffness found above is therefore an exact solution to the problem of an isotropic slab having the same axial force in the *x*- and *y*- direction. The expressions are very simple and by varying the point of zero stains ($k_2/(k_5h)$) from –0.5 to 0.5 one will get the stiffness as a function of the axial load. The curves are of course the same as the ones found from the numerical calculations.

Now, if one knows the applied loads and want to determine the stiffness one has to solve the 3rd-degree equation:

$$\begin{aligned} 0 &= 4\left(-\frac{k_2}{k_5 h}n_x - 3\frac{m_{xy}}{h}\right)\left(\frac{k_2}{k_5 h}\right)^2 + 3\left(n_x - (4+16\phi n)\frac{m_{xy}}{h}\right)\frac{k_2}{k_5 h} + n_x - 3\frac{m_{xy}}{h} \\ 0 &= \left(\frac{k_2}{k_5 h}\right)^3 + 3\frac{m_{xy}}{h n_x}\left(\frac{k_2}{k_5 h}\right)^2 + \left((3+12\phi n)\frac{m_{xy}}{h n_x} - \frac{3}{4}\right)\frac{k_2}{k_5 h} + \frac{3}{4}\frac{m_{xy}}{h n_x} - \frac{1}{4} \end{aligned} \tag{7.3.34}$$

The approach for solving this equation is of course the same as for bending.

7.3.3.1 Different axial forces

If the axial force in the two directions are different but still have the same sign numerical calculations show that the torsional stiffness, as a function of the degree of cracking, is the same.

An example may be seen in Figure 7.24 where results for the cases $n_y=2n_x$ and $n_y=n_x$ are plotted.

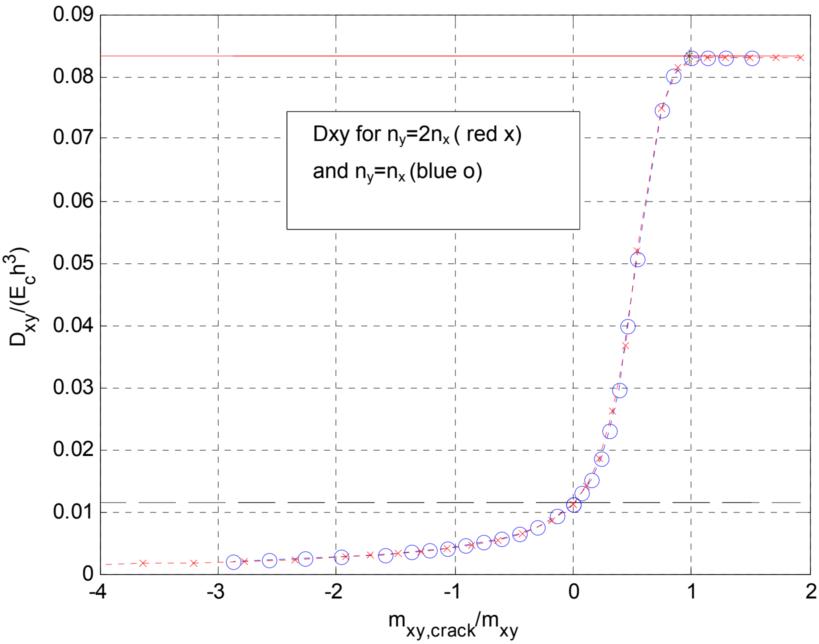


Figure 7.24 Torsional stiffness for a slab subjected to axial force in two directions, $n_y=2n_x$ (red x) and a slab subjected to axial force in two directions, $n_y=n_x$ (blue o) . The degree of isotropic reinforcement is $n\phi=0.05$ in both cases.

If the axial forces do not have the same sign it is not possible to define a degree of cracking and a simple approach has not been found. In this case one needs to carry out the numerical calculations for the given axial forces. Stiffnesses for different ratios of axial forces may be seen in Figure 7.25.

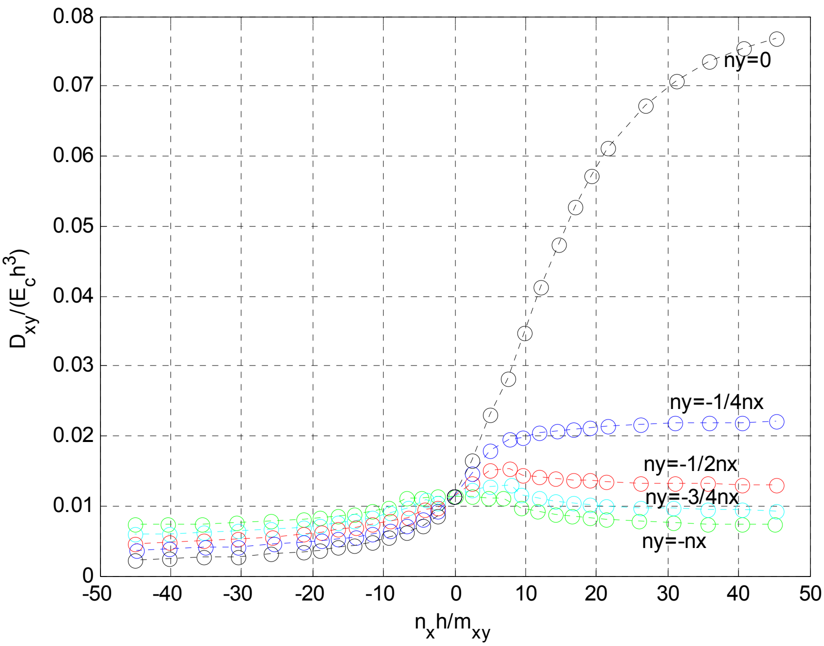


Figure 7.25 Torsional stiffness for a slab with different ratios of axial forces. The degree of isotropic reinforcement is nϕ=0.05.

7.3.3.2 Anisotropic reinforcement

Often the reinforcement in the *x*- and *y*- directions are different. For example it is more economical to let the direction of the shorter span in a rectangular slab supported along all edges carry more than the longer one. Thus it is interesting to investigate this case. It has been proposed, see [11], that in the case of no axial force the stiffness for an anisotropic slab may be calculated using the formulas for isotropic slabs and inserting an amount of reinforcement of:

$$A_s = \sqrt{A_{sx} A_{sy}} \tag{7.3.35}$$

Numerical calculations have shown that this assumption, equation (7.3.35), is correct. In Figure 7.26 it is seen that calculations on an anisotropic slab with a ratio of one to five between the reinforcements in the *x*- and *y*-direction lead to the same result as calculations on an isotropic slab with the reinforcement calculated according to (7.3.35).

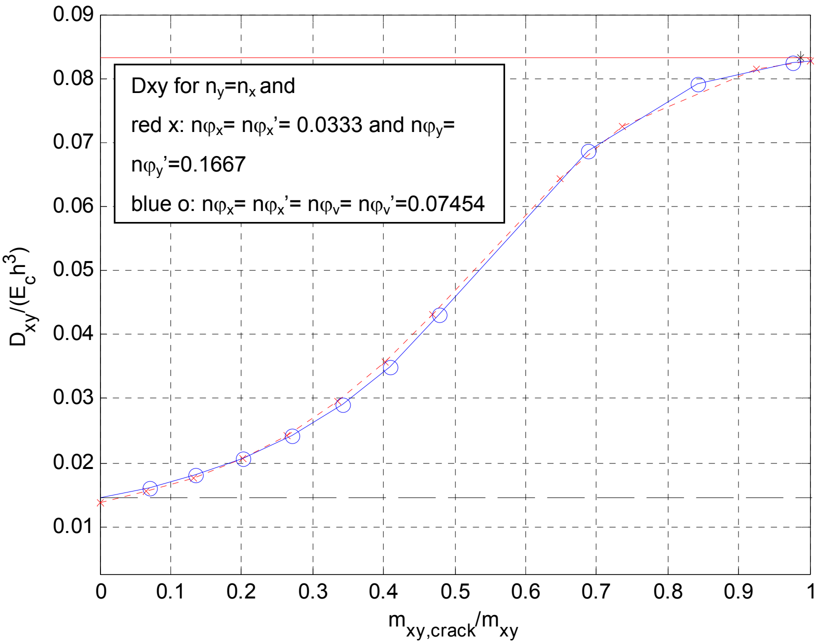


Figure 7.26 Torsional stiffness for slabs subjected to axial forces in two directions, n_y=n_x. Blue, solid line with o –markers is valid for an isotropic slab with $n\varphi_x$= $n\varphi_x$'= $n\varphi_y$= $n\varphi_y$'=0.07454 and red, dotted line with x- markers is valid for a slab with $n\varphi_x$= $n\varphi_x$'= 0.0333 and $n\varphi_y$= $n\varphi_y$'=0.1667.

7.3.4 Lower bound solutions

As for beams, we shall try to find lower bound stiffnesses for slabs subjected to torsion by assuming a statically admissible stress distribution and then use the virtual work equation.

If the concrete stresses are assumed to vary linearly from zero to σ_c over the depth a, as illustrated in Figure 7.27, we know that in the isotropic case with n_x=n_y we have the correct destribution.

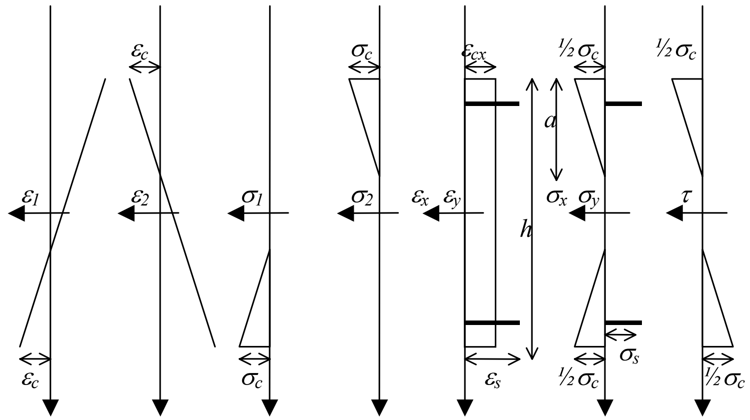


Figure 7.27 Stress destribution assumed in the lower bound solution for torsional stiffness.

In the isotropic case with n_x=n_y the principal strains are determined by an angle $\alpha = 45°$ which means that $\cos^2(\alpha)$=$\sin^2(\alpha)$=½. Thus we have the stresses illustrated in Figure 7.27.

The equilibrium equations become:

$$m_{xy} = \left(h - \frac{2}{3}a\right)\frac{1}{2}a\frac{1}{2}\sigma_c \tag{7.3.36}$$

$$m_x = m_y = 0 \tag{7.3.37}$$

$$n_x = n_y = 2\frac{1}{2}a\frac{1}{2}\sigma_c - 2\phi h \sigma_s \tag{7.3.38}$$

$$n_{xy} = 0 \tag{7.3.39}$$

σ_c may be determined from the moment equation and σ_s may be determined from the projection equation which lead to:

$$\sigma_c = \frac{12 m_{xy}}{h^2\left(3 - 2\frac{a}{h}\right)\frac{a}{h}} \tag{7.3.40}$$

$$\varepsilon_c = \frac{12 m_{xy}}{E_c h^2\left(3 - 2\frac{a}{h}\right)\frac{a}{h}} \tag{7.3.41}$$

$$\sigma_s = \frac{m_{xy}\left(-3\frac{n_x h}{m_{xy}} + 2\frac{n_x h}{m_{xy}}\frac{a}{h} + 6\right)}{2h^2\left(3 - 2\frac{a}{h}\right)\phi} \tag{7.3.42}$$

$$\varepsilon_s = \frac{m_{xy}\left(-3\frac{n_x h}{m_{xy}} + 2\frac{n_x h}{m_{xy}}\frac{a}{h} + 6\right)}{E_c n 2h^2\left(3 - 2\frac{a}{h}\right)\phi} \tag{7.3.43}$$

Applying a fictitious torsional moment of magnitude 1 and choosing the same stress distribution as determined above, we get:

$$\sigma_c{}^1 = \frac{12}{h^2\left(3 - 2\frac{a}{h}\right)\frac{a}{h}} \tag{7.3.44}$$

$$\sigma_s{}^1 = \frac{3}{h^2\left(3 - 2\frac{a}{h}\right)\phi} \tag{7.3.45}$$

It should be noted that when applying the fictitious moment we have no axial force. The work equation becomes:

$$2m_{xy}{}^1\kappa_{xy} = 2\frac{1}{3}a\sigma_c{}^1\varepsilon_c + 4\phi h\sigma_s{}^1\varepsilon_s \tag{7.3.46}$$

Solving this equation with respect to the geometrical torsion we get:

$$\kappa_{xy} = \frac{3m_{xy}\left(16\phi n - 3\frac{n_x h}{m_{xy}}\frac{a}{h} + 2\frac{n_x h}{m_{xy}}\left(\frac{a}{h}\right)^2 + 6\frac{a}{h}\right)}{\frac{a}{h}h^3\left(-3 + 2\frac{a}{h}\right)^2 E_c\phi n} \tag{7.3.47}$$

From this we may calculate the torsional stiffness as:

$$D_{xy} = \frac{m_{xy}}{\kappa_{xy}} \Rightarrow$$

$$\frac{D_{xy}}{E_c h^3} = \frac{1}{3}\frac{\frac{a}{h}\left(3 - 2\frac{a}{h}\right)^2\phi n}{\left(16\phi n - 3\frac{n_x h}{m_{xy}}\frac{a}{h} + 2\frac{n_x h}{m_{xy}}\left(\frac{a}{h}\right)^2 + 6\frac{a}{h}\right)} \tag{7.3.48}$$

It should be noted that setting the axial force to zero and using the compatibility conditions to determine *a/h*, as described by equation (7.3.16), we get the exact solution.

In Figure 7.28 results of the lower bound solution for the value of *a* leading to the largest stiffness are shown along with the exact solution from the numerical calculations. Figure 7.29 shows the values of *a/h* used in the calculations of the stiffness in Figure 7.28. The values of *a/h* are found by differentiation of formula (7.3.48) with respect to *a/h*, setting this expression equal to zero and then solving with respect to *a/h*. The expression for *a/h* is quite extensive and will not be given here.

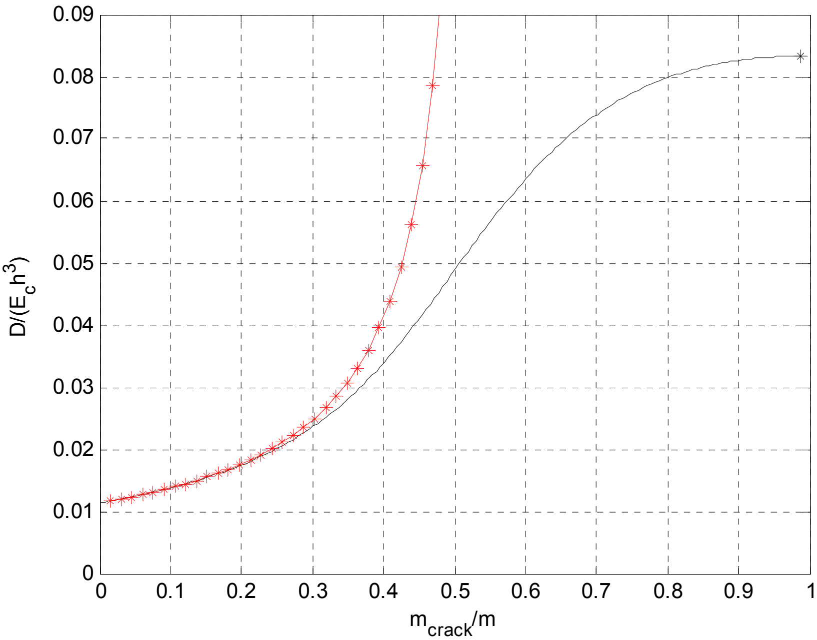


*Figure 7.28Torsional stiffness for s slab found by formula (7.3.48) (red *) and the exact solution. It is assumed that nφ= nφ'=0.05.*

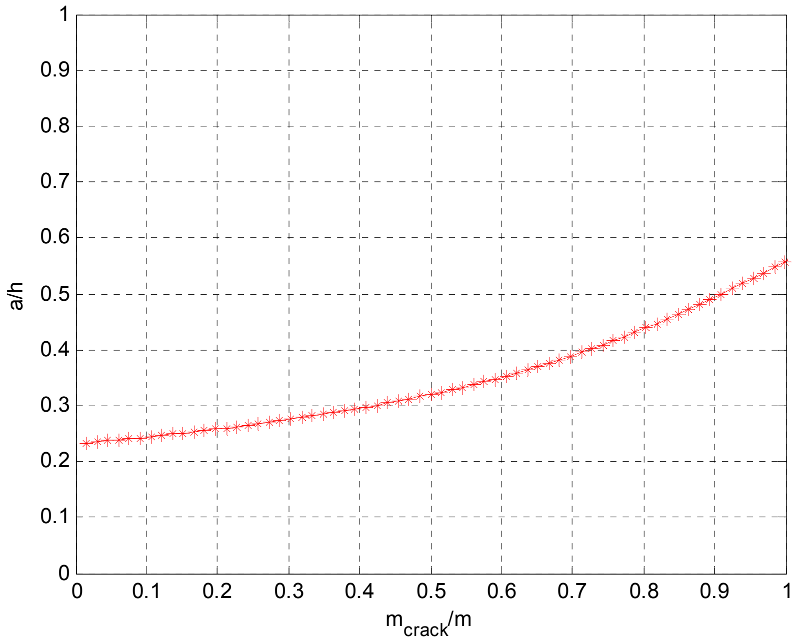


Figure 7.29a/h used in the calculations I Figure 7.28.

It appears that the stiffnesses found are sometimes higher than the exact stiffness and therefore they can not be true lower bound solutions. Evaluating the stiffness from formula (7.3.48) it appears that there is a point where the stiffness becomes infinitely large.

This leads to the important conclusion that the stiffness found using the work equation does not always lead to a true lower bound solutions. The reason is that in this case we are dealing with not only one single stiffness constant but with stiffnesses for curvature as well as for stretching.

Another approach that can be used in cases of several stiffnesses is to minimize the complementary energy with respect to *a/h*. If this method is used on the stress field described above it leads to the exact solution.

7.3.5 Torsional stiffness from disk solutions

Another way of approaching the problem is to subdivide the slab into two disks at top and bottom and then calculate the stiffness from the formulas for disk stiffness. If equilibrium is satisfied we have a lower bound solution since the compatibility conditions are disregarded.

Determination of the torsional stiffness for an isotropic slab with to no axial forces leads to the following calculations:

For an isotropic disk in the cracked state we have, see [11]:

$$\tau = \frac{\frac{E_c}{2}\varphi}{2+\frac{1}{n\rho_{disk}}} \Leftrightarrow \varphi = \frac{\tau\left(2+\frac{1}{n\rho_{disk}}\right)}{\frac{E_c}{2}} \tag{7.3.49}$$

where φ is the change of angle and ρ_{disk} is the reinforcement ratio for the fictitious disk. Assuming that the disks have the thickness a and the slab has the depth h we find:

$$\rho_{disk} = \frac{\phi h}{a} \tag{7.3.50}$$

The equilibrium equations give:

$$m_{xy} = \tau\left(h-a\right)a \tag{7.3.51}$$

where τ is the shear stress in the disks.

Using the work equation we find:

$$m_{xy}{}^{1} = 1 \Rightarrow \tau^{1} = \frac{1}{\left(h-a\right)a}$$

$$m_{xy}{}^{1}\kappa_{xy} = \tau^{1}\varphi a$$

$$1\kappa_{xy} = \frac{1}{\left(h-a\right)}\frac{\tau\left(2+\frac{1}{n\rho_{disk}}\right)}{\frac{E_c}{2}}$$

$$\kappa_{xy} = \frac{\tau\left(2+\frac{1}{n\rho_{disk}}\right)}{\frac{E_c}{2}\left(h-a\right)}$$

$$D_{xy} = \frac{m_{xy}}{\kappa_{xy}} = \frac{1}{2}\frac{\left(h-a\right)^2 aE_c}{2+\frac{1}{n\rho_{disk}}} \tag{7.3.52}$$

The optimum solution for the stiffness may be found by differentiating with respect to a and inserting the result into the solution for the stiffness:

$$\frac{dD_{xy}}{da} = 0 \Rightarrow a = \frac{1}{2}\left(3n\phi - \sqrt{n\phi\left(9n\phi+4\right)}\right)h \tag{7.3.53}$$

$$\frac{D_{xy}}{E_c h^3} = \frac{\left(2+3n\phi-\sqrt{n\phi\left(9n\phi+4\right)}\right)^2\left(3n\phi-\sqrt{n\phi\left(9n\phi+4\right)}\right)n\phi}{8\left(n\phi+\sqrt{n\phi\left(9n\phi+4\right)}\right)} \tag{7.3.54}$$

This solution and the exact solution may be compared for specific values of ϕn. Setting n=10 and ϕ=1% lead to a stiffness $D_{xy}/(E_c h^3)$ = 0.016 whereas the exact stiffness according to (7.3.17) becomes $D_{xy}/(E_c h^3)$ = 0.0169. Thus the agreement is very good. Disk solutions may also be found for slabs subjected to axial forces as well.

7.3.6 The relation between bending stiffness and torsional stiffness

It is of course interesting to look for a relation between the bending stiffness and the torsional stiffness. An approximation formula for different reinforcement arrangements has been given in the literature and has also been used for verification purposes in section 7.3.3.2. Nevertheless, the formula has never been verified in the cracked state with axial forces.

The suggested formula for the torsional stiffness expressed by the bending stiffness, see [11],[5],[6], is:

$$D_{xy} = \frac{1}{2}\sqrt{D_x D_y} \tag{7.3.55}$$

In [11] it is argued that this approximation leads to good agreement if there is no axial force and if the effective depth, *d,* equals the depth of the slab, *h*. Nevertheless, it is evident that in the uncracked state this formula lead to a torsional stiffness of only half the bending stiffness instead of giving almost the same stiffness as the bending stiffness. Therefore, the formula can not be general.

The fact that the formula can not be general may also be demonstrated in the following way:

One of the differences in the stiffness calculations for bending and torsion is the effect of the position of the reinforcement. Assuming that the reinforcement is placed symmetrically, the bending stiffness increases when the reinforcement is placed more and more close to the faces while the torsional stiffness remains constant. This fact also rules out any general relation between the stiffnesses.

Nevertheless, we may show that there is a way of calculating the torsional stiffness from the formulas of bending stiffness. Using the knowledge about the effect of the position of the reinforcement it is evident that we must lock the position of the reinforcement in the bending situation in order to reach the possibility of a general agreement. Knowing that in the uncracked case the torsional stiffness is the same as the bending stiffness for an unreinforced slab the locked position must be the centre of the slab. Here we keep in mind that in the uncracked case the bending stiffness of an unreinforced slab is the same as for a reinforced slab with the reinforcement in the centre.

If the torsional stiffness is calculated from the bending stiffness of a slab where all the reinforcement is “moved” to the centre we shall have agreement in the uncracked state as well as for all positions of the reinforcement. Then only the amount of reinforcement and the applied axial force to be used in the calculation has to be determined.

Considering the case of no axial force, formula (7.2.3) may be used to determine the bending stiffness of the fictitious slab by setting the effective depth, *d*, equal to half the depth, *h*. In this case we get:

$$\frac{D_b}{E_c b h^3} = \frac{1}{8}\begin{pmatrix} \frac{2}{3}(n\phi)^3 + 2(n\phi)^2 + n\phi \\ -\frac{2}{3}(n\phi)^2\sqrt{(n\phi)^2 + 2n\phi} - \frac{4}{3}n\phi\sqrt{(n\phi)^2 + 2n\phi} \end{pmatrix} \tag{7.3.56}$$

The torsional stiffness calculated according to formula (7.3.17) is:

$$\frac{D_t}{E_c h^3} = \begin{pmatrix} \frac{2}{3}(n\phi)^3 + (n\phi)^2 + \frac{1}{4}n\phi \\ -\frac{2}{3}(n\phi)^2\sqrt{(n\phi)^2 + 2n\phi} - \frac{2}{3}n\phi\sqrt{(n\phi)^2 + 2n\phi} \end{pmatrix} \tag{7.3.57}$$

From these two formulas it seems that the influence of the reinforcement is not the same. However, if we assume that the reinforcement ratio applied when calculating the fictitious bending stiffness is twice the reinforcement ratio applied for the torsional stiffness calculation we find that the formulas are the same. Formula (7.3.56) is valid for one layer of reinforcement and therefore one may say that they are the same if all reinforcement is moved to the centre.

It is hereby shown that in the case of no axial force and in the uncracked slab the torsional stiffness is the same as the bending stiffness of a slab with twice the reinforcement placed in the centre.

More generally it may be shown that an isotropic slab, with the reinforcement placed in the centre and subjected to the same axial force in both directions, has the same torsional and bending stiffness. This is easily seen by considering the equilibrium equations as demonstrated in Appendix 10.3.

Numerical calculations also confirm that this approach leads to perfect agreement.

Figure 7.30 shows the bending stiffness for a slab with $n\phi$=0.10 (one layer of reinforcement) as a function of the degree of cracking, and the torsional stiffness for a slab with $n\phi$=0.05 as a function of the degree of cracking. Figure 7.30 is a combination of Figure 7.18 and Figure 7.21 into the same graph.

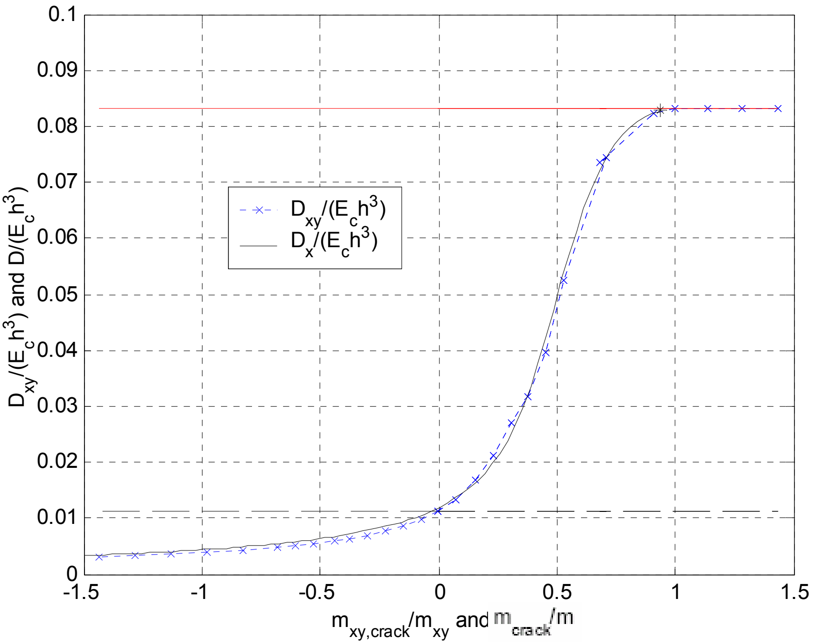
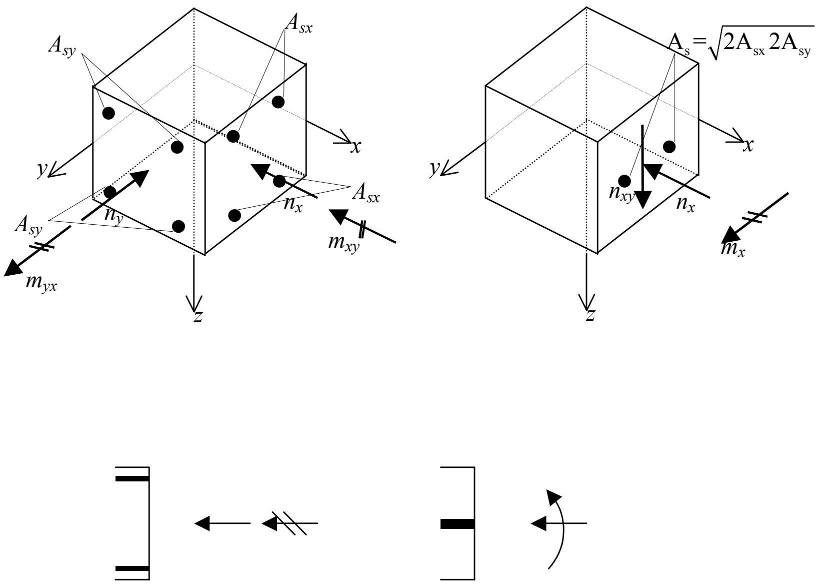


Figure 7.30Bending stiffness and torsional stiffness compared.

It is seen from Figure 7.30 that besides from the numerical deviations the graphs are the same. This means that the suggested approach may be used in general. The approach facilitates the calculations of the torsional stiffness a great deal.

Figure 7.31 illustrates the relation between torsional- and bending stiffness in general.



$$\frac{D_{xy}\left(\frac{m_{xy,cracked}}{m_{xy}}\right)}{h^3 E_c} = \frac{D_x\left(\frac{m_{x,cracked}}{m_x}\right)}{h^3 E_c}$$

Figure 7.31. The relation between torsional- and bending stiffness for an isotropic slab.

8 Conclusion

In this paper it is shown how to calculate the stiffnesses of cracked concrete slabs in general using an assumption of linear elastic behaviour of both reinforcement and concrete and of no tensile strength of the concrete.
Any slab stiffness may be found by the numerical method described in this paper. However, the main result of this investigation is the determination of the torsional stiffness of a slab subjected to axial forces.
If the axial forces have the same sign the torsional stiffness for a certain degree of cracking is the same as the bending stiffness for the same degree of cracking if the reinforcement is placed in the centre. Since the torsional stiffness is independent of the position of the reinforcement, as long as it is placed symmetrically about the centre, the torsional stiffness may be calculated from the formulas for the bending stiffness.
If the axial forces do not have the same sign, no simple method has been found and in this case it is necessary to determine the torsional stiffness using the numerical methods.

The use of lower bound methods has also been investigated. For bending such solutions may be useful for practical purposes. For torsion and axial force simple lower bound solutions are shown to be too inaccurate.

9 Literature

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10 Appendix

10.1 The depth of the compression zone, y_0, for bending with axial force.

The problem, given in section 7.3.1 in (7.3.5), is:

$$
\begin{aligned}
0 = & \left(\frac{y_0}{h}\right)^3 + \left(\frac{3}{\frac{n_x h}{m_x}} - \frac{3}{2}\right)\left(\frac{y_0}{h}\right)^2 \\
& + \left(\frac{6(n\phi' + n\phi)}{\frac{n_x h}{m_x}} - 6\left(n\phi\left(\frac{h_c}{h} - \frac{1}{2}\right) + n\phi'\left(\frac{1}{2} - \frac{h_c'}{h}\right)\right)\right)\frac{y_0}{h} \\
& +6\left(\frac{n\phi\left(\frac{h_c}{h} - 1\right) + n\phi'\left(-\frac{h_c'}{h}\right)}{\frac{n_x h}{m_x}}\right) - 6\left(n\phi\left(\frac{1}{2} - \frac{3}{2}\frac{h_c}{h} + \left(\frac{h_c}{h}\right)^2\right) + n\phi'\left(-\frac{1}{2}\frac{h_c'}{h} + \left(\frac{h_c'}{h}\right)^2\right)\right)
\end{aligned}
\tag{10.1.1}
$$

Writing this equation in a general form leads to the following coefficients:

$$0 = \left(\frac{y_0}{h}\right)^3 + a_1\left(\frac{y_0}{h}\right)^2 + a_2\frac{y_0}{h} + a_3$$

where

$$a_1 = \left(\frac{3}{\frac{n_x h}{m_x}} - \frac{3}{2}\right)$$

$$a_2 = \left(\frac{6(n\phi' + n\phi)}{\frac{n_x h}{m_x}} - 6\left(n\phi\left(\frac{h_c}{h} - \frac{1}{2}\right) + n\phi'\left(\frac{1}{2} - \frac{h_c'}{h}\right)\right)\right)$$

$$a_3 = 6\left(\frac{n\phi\left(\frac{h_c}{h} - 1\right) + n\phi'\left(-\frac{h_c'}{h}\right)}{\frac{n_x h}{m_x}}\right) - 6\left(n\phi\left(\frac{1}{2} - \frac{3}{2}\frac{h_c}{h} + \left(\frac{h_c}{h}\right)^2\right) + n\phi'\left(-\frac{1}{2}\frac{h_c'}{h} + \left(\frac{h_c'}{h}\right)^2\right)\right)$$

(10.1.2)

The solution is:

$$\frac{y_0}{h} = S + T - \frac{1}{3}a_1$$

where

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}$$

$$T = \sqrt[3]{R - \sqrt{Q^3 + R^2}} \qquad (10.1.3)$$

$$Q = \frac{3a_2 - 3a_1^2}{9}$$

$$R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}$$

10.2 Description of a general program for the determination of the strains in a section

The program has the structure described below:

1. The geometrical and physical properties are given (h, h_{cx}, h_{cx}' h_{cy}, h_{cy}', A_{sx}, A_{sx}', A_{sy}, A_{sy}', E_s, E_c).
2. The applied forces and moments are given (n_x , n_y, n_{xy}, m_x, m_y, m_{xy}).

3. An estimate of the six constants k_1 to k_6 is made. These are combined into a vector **K**

$$\mathbf{K} = \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{Bmatrix}$$

4. An acceptable deviation is set. This deviation is defined as

$$dev = \frac{\frac{\left|m_x - m_x^*\right|}{h} + \frac{\left|m_y - m_y^*\right|}{h} + \frac{\left|m_{xy} - m_{xy}^*\right|}{h} + \left|n_x - n_x^*\right| + \left|n_y - n_y^*\right| + \left|n_{xy} - n_{xy}^*\right|}{\frac{\left|m_x\right|}{h} + \frac{\left|m_y\right|}{h} + \frac{\left|m_{xy}\right|}{h} + \left|n_x\right| + \left|n_y\right| + \left|n_{xy}\right|}$$

where the upper index * indicates the results for a given estimate of **K.**

5. The number of sections into which the depth is subdivided when making a numerical integration is set.
6. A loop is started and this loop runs while the deviation is larger than the deviation, set in 4.
   a. The applied forces and moments corresponding to **K** is calculated (n_x^*, n_y^*, n_{xy}^*, m_x^*, m_y^*, m_{xy}^*).
   b. A loop is started that runs 6 times varying one of the k values at a time from k to k^+ and each time calculating the applied forces and moments (upper index +). Thereby a difference operator called **JJ** is established. **JJ** is given by:

$$\mathbf{JJ} = \begin{Bmatrix}
\frac{k_1^+ - k_1}{m_x^+ - m_x} & \frac{k_1^+ - k_1}{n_x^+ - n_x} & \frac{k_1^+ - k_1}{m_y^+ - m_y} & \frac{k_1^+ - k_1}{n_y^+ - n_y} & \frac{k_1^+ - k_1}{m_{xy}^+ - m_{xy}} & \frac{k_1^+ - k_1}{n_{xy}^+ - n_{xy}} \\
\frac{k_2^+ - k_2}{m_x^+ - m_x} & \frac{k_2^+ - k_2}{n_x^+ - n_x} & \frac{k_2^+ - k_2}{m_y^+ - m_y} & \frac{k_2^+ - k_2}{n_y^+ - n_y} & \frac{k_2^+ - k_2}{m_{xy}^+ - m_{xy}} & \frac{k_2^+ - k_2}{n_{xy}^+ - n_{xy}} \\
\frac{k_3^+ - k_3}{m_x^+ - m_x} & \frac{k_3^+ - k_3}{n_x^+ - n_x} & \frac{k_3^+ - k_3}{m_y^+ - m_y} & \frac{k_3^+ - k_3}{n_y^+ - n_y} & \frac{k_3^+ - k_3}{m_{xy}^+ - m_{xy}} & \frac{k_3^+ - k_3}{n_{xy}^+ - n_{xy}} \\
\frac{k_4^+ - k_4}{m_x^+ - m_x} & \frac{k_4^+ - k_4}{n_x^+ - n_x} & \frac{k_4^+ - k_4}{m_y^+ - m_y} & \frac{k_4^+ - k_4}{n_y^+ - n_y} & \frac{k_4^+ - k_4}{m_{xy}^+ - m_{xy}} & \frac{k_4^+ - k_4}{n_{xy}^+ - n_{xy}} \\
\frac{k_5^+ - k_5}{m_x^+ - m_x} & \frac{k_5^+ - k_5}{n_x^+ - n_x} & \frac{k_5^+ - k_5}{m_y^+ - m_y} & \frac{k_5^+ - k_5}{n_y^+ - n_y} & \frac{k_5^+ - k_5}{m_{xy}^+ - m_{xy}} & \frac{k_5^+ - k_5}{n_{xy}^+ - n_{xy}} \\
\frac{k_6^+ - k_6}{m_x^+ - m_x} & \frac{k_6^+ - k_6}{n_x^+ - n_x} & \frac{k_6^+ - k_6}{m_y^+ - m_y} & \frac{k_6^+ - k_6}{n_y^+ - n_y} & \frac{k_6^+ - k_6}{m_{xy}^+ - m_{xy}} & \frac{k_6^+ - k_6}{n_{xy}^+ - n_{xy}}
\end{Bmatrix}$$

c. A vector **F** containing the differences in the applied forces and moments is calculated:

$$\mathbf{F} = \begin{Bmatrix} m_x^* - m_x \\ n_x^* - n_x \\ m_y^* - m_y \\ n_y^* - n_y \\ m_{xy}^* - m_{xy} \\ n_{xy}^* - n_{xy} \end{Bmatrix}$$

d. New values of **K** is calculated as

$$\mathbf{K}_{new} = \mathbf{K} - \left(\mathbf{J}\mathbf{J}^T\right)^{-1}\mathbf{F}$$

e. The deviation is calculated and an evaluation of this decides whether the loop starts again or not.

7. The stiffnesses are calculated from the curvatures determined by **K** and the applied forces and moments.

10.3 Torsional and bending stiffness for a slab with the reinforcement placed in the centre

In this appendix we consider a slab with the isotropic reinforcement placed in the centre. The reinforcement ratio is 2ϕ. The slab is subjected to the same axial force in both the *x*- and *y*- direction.

By comparing the equations for the bending stiffness and the torsional stiffness we may show that the stiffnesses are the same.

Generally the stiffnesses are defined by:

$$D_x = \frac{m_x}{\kappa_x} \qquad D_{xy} = \frac{m_{xy}}{\kappa_{xy}} \tag{10.1.4}$$

By showing that the moments and the curvatures have the same dependency on the axial force it is demonstrated that the stiffnesses are the same.

The moments are (see (7.3.2) and (7.3.36)):

$$m_x = \frac{1}{2}\sigma_c y_0\left(\frac{h}{2} - \frac{1}{3}\frac{h}{2}\right) = \frac{1}{12}h^2\left(3-2\frac{a}{h}\right)\frac{a}{h}\sigma_c$$
$$m_{xy} = \left(h - \frac{2}{3}a\right)\frac{1}{2}a\frac{1}{2}\sigma_c = \frac{1}{12}h^2\left(3-2\frac{y_0}{h}\right)\frac{y_0}{h}\sigma_c \tag{10.1.5}$$

The curvatures are:

$$\kappa_x = \frac{\frac{\sigma_c}{E_c}}{y_0}$$
$$\kappa_1 = \kappa_2 = \kappa_{xy} = \frac{\frac{\sigma_c}{E_c}}{a} \tag{10.1.6}$$

The projection equation for the bending case is (see (7.3.1)):

$$n_x = \frac{1}{2}\sigma_c y_0 - \left(\frac{\frac{\sigma_c}{E_c}}{y_0} nE_c\left(h - y_0 - \frac{h}{2}\right)\right)\phi h - \left(-\frac{\frac{\sigma_c}{E_c}}{y_0} nE_c\left(y_0 - \frac{h}{2}\right)\right)\phi' h \Leftrightarrow$$
$$n_x = \frac{1}{2}h\sigma_c \frac{\left(\left(\frac{y_0}{h}\right)^2 - 2\phi n + 4\phi n\frac{y_0}{h}\right)}{\frac{y_0}{h}} \tag{10.1.7}$$

For the torsion case we have the following formula for the stresses when the compatibility conditions are fulfilled:

$$\sigma_s = \left(\frac{\frac{1}{2}\sigma_c}{E_c} - \frac{\frac{\frac{1}{2}\sigma_c}{E_c}}{a}(h-a)\right)E_c n \tag{10.1.8}$$

This leads to the following projection equation (see (7.3.38):

$$n_x = 2\frac{1}{2}a\frac{1}{2}\sigma_c + 2\phi h\sigma_s \Leftrightarrow$$
$$n_x = \frac{1}{2}h\sigma_c \frac{\left(\left(\frac{a}{h}\right)^2 - 2\phi n + 4\phi n\frac{a}{h}\right)}{\frac{a}{h}} \tag{10.1.9}$$

It is seen that in both cases we have the same relation between moment, axial force, compression zone, maximum stress, curvature and therefore also stiffness. It is hereby shown that the torsional stiffness is the same as the bending stiffness if the reinforcement is isotropic and placed in the centre.

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